2210-90 Exam 2

Fall 2012

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Please try to do all all work in the space provided. Page 6 is blank in case you need extra paper. Please circle your final answer.

Small mistake -1
Grade following problems assuming previous ans.

1. (19pts) For this problem, consider the function

$$f(x,y) = xe^y + x^2 - 3xy.$$

(a) (3pts) Find $f_x(x,y)$

$$f_x(x,y) = e^y + 2x - 3y$$

(b) (3pts) Find $f_y(x,y)$

$$f_y(x,y) = xe^y - 3x$$

(c) (3pts) Find $\nabla f(1,0)$, that is, find the gradient of f at $(1,0)$.

$$\nabla f(1,0) = \langle f_x(1,0), f_y(1,0)\rangle = \langle 3, -2\rangle$$

(d) (3pts) Find the maximum rate of change of f at $(1,0)$.

$$\|\nabla f(1,0)\| = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$

(e) (3pts) Let $\mathbf{u} = \langle \frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\rangle$. Find $D_{\mathbf{u}}f(1,0)$, that is, find the directional derivative of f at $(1,0)$ in the direction of $\mathbf{u}$.

$$D_u f(1,0) = \nabla f(1,0) \cdot \langle \frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\rangle = \langle 3,-2\rangle \cdot \langle \frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\rangle = \sqrt{3} - \frac{2\sqrt{6}}{3}$$

(f) (4pts) Find the equation of the tangent plane to the graph of f at $(1,0)$.

$$f(1,0) = 2$$

$$z - 2 = 3(x-1) - 2y$$

2. (20pts) Evaluate the following limits, derivatives, or integrals. **Be sure to show your work.** Remember, a limit might not exist, in which case, write 'DNE' as your answer.

Correct answer w/ no work $+2$

Correct answer with incorrect reasoning -1

(a) (4pts) $\lim_{(x,y)\to(0,0)} \frac{x^3+2y^3}{x^2+y^2}$ (**Hint:** use polar coordinates)

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+2y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3(\cos^3\theta + 2\sin^3\theta)}{r^2} = \lim_{r\to 0} r(\cos^3\theta + 2\sin^3\theta) = 0$$

Small mistake -1

(b) (4pts) $\lim_{(x,y)\to(0,0)} \frac{x^2-2y^2}{x^2+y^2}$ $= \text{DNE}$

Why? when $(x,y)\to(0,0)$ along x-axis $(y=0)$, $\frac{x^2-2y^2}{x^2+y^2} = 1$,

but when $(x,y)\to(0,0)$ along y-axis $(x=0)$,

$\frac{x^2-2y^2}{x^2+y^2} = -2$. So limit is not independent of path to $(0,0)$.

(c) (4pts) Find $\frac{\partial^2 z}{\partial x \partial y}$ when $z = y\sin(xy)$.

$$\frac{\partial z}{\partial y} = \sin(xy) + yx\cos(xy)$$

$$\frac{\partial z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\sin(xy) + yx\cos(xy)\right) = 2y\cos(xy) - y^2 x\sin(xy)$$

(d) (4pts) $\int_0^2 \int_0^1 xy^2 \, dx \, dy$

$$= \int_0^2 \left(\frac{x^2y^2}{2} \Big|_{0=x}^{1=x} \right) dy = \int_0^2 \frac{y^2}{2} \, dy = \left(\frac{y^3}{6} \Big|_0^2 = \frac{8}{6} = \frac{4}{3}$$

(e) (4pts) $\int_0^2 \int_0^1 (x+y^2) \, dx \, dy$

$$= \int_0^2 \left(\frac{x^2}{2} + y^2 x \Big|_{0=x}^{1=x} \right) dy = \int_0^2 \left(\frac{1}{2} + y^2\right) dy = \left(\frac{1}{2}y + \frac{y^3}{3} \Big|_0^2$$

$$= 1 + \frac{8}{3} = \frac{11}{3}$$

3. (8pts) A bullet is manufactured with an intended mass of .1 kg, but errors in the manufacturing process result in bullets which can be too light or too heavy by as much as .001 kg. One such bullet is fired from a gun and observed to have a velocity of 1000 m/s, give or take about 20 m/s. Use differentials to estimate the possible error in the calculation of the kinetic energy of the bullet $K = \frac{1}{2}mv^2$. **Note:** Energy has units of joules (J); J=kg·m^2/s^2.

$K(m,v) = \frac{1}{2}mv^2$

$m = .1\text{kg}$
$dm = .001 \text{ kg}$
$v = 1000 \text{ m/s}$
$dv = 20 \text{ m/s}$

2

$dK = K_m\, dm + K_v\, dv$

$= \frac{1}{2}v^2 dm + mv\,dv$ 4

$= \frac{1}{2}(1000)^2(.001) + (.1)(1000)(20)$

$= 500 + 2000 = 2500 \text{ J}$

2

4. (13pts) Consider the function

$$f(x,y) = x^2y - y - 2x.$$

(a) (5pts) Find the 2 critical points of f.

2

$\langle 0,0\rangle = \nabla f = \langle 2xy-2,\ x^2-1\rangle$

So $0 = 2xy - 2$
$0 = x^2 - 1 \Rightarrow x = \pm 1.$

when $x=1$,
$0 = 2y - 2 \Rightarrow y = 1.$

when $x=-1$
$0 = -2y - 2 \Rightarrow y = -1.$

3

Critical pts: $(1,1)$ & $(-1,-1)$

(b) (4pts) Find the discriminant, $D = f_{xx}f_{yy} - (f_{xy})^2$.

$f_{xx} = 2y$ 2
$f_{xy} = 2x$
$f_{yy} = 0$

$D = (2y)(0) - (2x)^2 = -4x^2 < 0.$

2

(c) (4pts) Use the discriminant to determine whether each of the critical points found in part (a) is a local minimum, a local maximum, or a saddle point.

Since $D < 0$, both critical pts are saddle pts.

5. (12pts) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 - y^2$ on the ellipse $9x^2 + y^2 = 9$.

$$\nabla f = \lambda \nabla g$$

$$\langle 2x, -2y \rangle = \lambda \langle 18x, 2y \rangle$$

We get 3 equs: 6

$$2x = 18\lambda x \quad ①$$
$$-2y = 2\lambda y \quad ②$$
$$9x^2 + y^2 = 9 \quad ③$$

Now ② $\Rightarrow \lambda = -1$ or $y = 0$.

if $\lambda = -1$, ① $\Rightarrow 2x = -18x \Rightarrow x = 0$.

which gives pts $(0,3)$ & $(0,-3)$.

if $y = 0$, then $x = \pm 1$, which gives pts 4

$(1,0)$ & $(-1,0)$.

Now we plug in:

$f(0,3) = f(0,-3) = -9$ min

$f(1,0) = f(-1,0) = 1$ max

2

6. (8pts) The equation

$$2xz + y^2z^2 = 3$$

defines z implicitly as a function of x and y.

(a) (4pts) Find $\frac{\partial z}{\partial x}$ at $(1, -1, 1)$.

$$2z + 2x\frac{\partial z}{\partial x} + 2y^2 \frac{\partial z}{\partial x} \cdot z = 0 \quad 2$$

$$2 + 2\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{1}{2} \quad 2$$

(b) (4pts) Find $\frac{\partial z}{\partial y}$ at $(1, -1, 1)$.

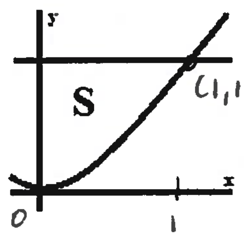
$$2x\frac{\partial z}{\partial y} + 2yz^2 + 2y^2 z\frac{\partial z}{\partial y} = \cancel{3}\,0 \quad 2$$

$$2\frac{\partial z}{\partial y} - 2 + 2\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{1}{2} \quad 2$$

7. (10pts) Find

$$\iint_S (x+2y)\ dA,$$

where S is the region bounded by $x=0$, $y=x^2$, and $y=1$. **Note:** Below is a rough sketch of the region S.



5

wrong limits -4

$$\iint_S (x+2y)\,dA = \int_0^1 \int_{x^2}^1 (x+2y)\,dy\,dx$$

$$= \int_0^1 \left(xy + y^2 \Big|_{x^2=y}^{1=y} \right) dx$$

$$= \int_0^1 \left(x+1-x^3-x^4 \right) dx$$ 2

$$= \left(\frac{x^2}{2} + x - \frac{x^4}{4} - \frac{x^5}{5} \Big|_0^1 \right.$$ 3

$$= \frac{1}{2}+1-\frac{1}{4}-\frac{1}{5} = \frac{10}{20}+\frac{20}{20}-\frac{5}{20}-\frac{4}{20} = \frac{21}{20}$$

or

$$\int_0^1 \int_0^{\sqrt{y}} (x+2y)\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^2}{2} + 2xy \Big|_{x=0}^{x=\sqrt{y}} \right) dy$$

$$= \int_0^1 \left(\frac{y}{2} + 2y^{3/2} \right) dy = \left(\frac{y^2}{4} + \frac{4}{5} y^{5/2} \Big|_0^1 \right.$$

$$= \frac{1}{4} + \frac{4}{5} = \frac{21}{20}$$

8. (10pts) Using polar coordinates, compute

$$\iint_D e^{-x^2-y^2}\ dA,$$

where D is the disk $x^2+y^2 \le 9$.

5 missing $r=-3$

$$\iint_D e^{-x^2-y^2}\,dA = \int_0^{2\pi} \int_0^3 e^{-r^2} r\,dr\,d\theta$$

$$= 2\pi \int_0^3 e^{-r^2} r\,dr = 2\pi \left(-\tfrac{1}{2}\right) \int_0^{-9} e^u\,du$$

Set $u=-r^2$ 2

$du = -2r\,dr \Rightarrow r\,dr = -\frac{1}{2}du$

$$= -\pi \left(e^u \Big|_0^{-9} \right.$$

$$= -\pi \left(e^{-9} - 1 \right)$$

$$= \pi \left(1 - e^{-9} \right)$$

3