2210-90 Exam 1
Summer 2014

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Page 5 is blank in case you need extra paper. Please circle your final answers.

1. (14pts) Consider the vectors $\mathbf{u} = \langle -1, 5, 2\rangle$ and $\mathbf{v} = \langle 2, 2, -3\rangle$. Find

   (a) (2pts) $2\mathbf{u} - 3\mathbf{v}$

   2

   $2\langle -1,5,2\rangle - 3\langle 2,2,-3\rangle = \langle -2,10,4\rangle - \langle 6,6,-9\rangle = \langle -8,4,13\rangle$

   (b) (2pts) $||\mathbf{u}||$

   2

   $||\bar{u}|| = \sqrt{(-1)^2+5^2+2^2} = \sqrt{1+25+4} = \sqrt{30}$

   (c) (2pts) $\mathbf{u}\cdot\mathbf{v}$

   2

   $\bar{u}\cdot\bar{v} = (-1)(2) + (5)(2) + (2)(-3) = -2+10-6 = 2$

   (d) (4pts) the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$. Recall, this is the dot product of $\mathbf{u}$ with the unit vector pointing in the same direction as $\mathbf{v}$.

   4

   $$\frac{\bar{u}\cdot\bar{v}}{||\bar{v}||} = \frac{2}{\sqrt{2^2+2^2+3^2}} = \frac{2}{\sqrt{17}}$$

   (e) (4pts) $\mathbf{u}\times\mathbf{v}$

   4

   $$\bar{u}\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 5 & 2 \\ 2 & 2 & -3 \end{vmatrix} = \hat{i}(-15-4) + \hat{j}(4-3) + \hat{k}(-2-10)$$

   $$= -19\hat{i} + \hat{j} - 12\hat{k} = \langle -19, 1, -12\rangle$$

2. (8pts) Find an equation of the plane containing the point $(0, 5, -4)$ which is perpendicular to the line

   $$\mathbf{r}(t) = \langle 1, -1, 4\rangle + t\langle 3, -2, -1\rangle$$

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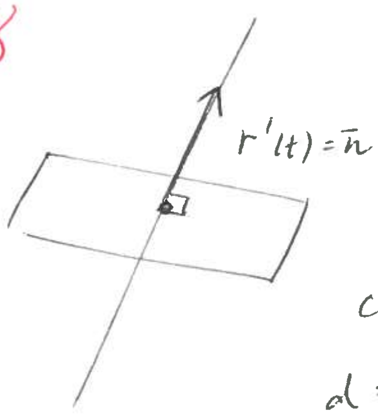

$\bar{n} = \langle 3, -2, -1\rangle$.

$3x - 2y - z = d$

contains the point $(0,5,-4)$

$d = 3(0) - 2(5) - (-4) = 0 - 10 + 4 = -6$.

$3x - 2y - z = -6$

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3. (6pts) Find the equation of the largest sphere centered at $(2,3,5)$ that is completely contained in the first octant. **Note:** the first octant is where $x \geq 0$, $y \geq 0$, and $z \geq 0$.

radius can be at most 2, since any larger radius would have pts with $x < 0$.

6

$$(x-2)^2 + (y-3)^2 + (z-5)^2 = 2^2 = 4.$$

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

4. (17pts) Suppose a particle's position at time t is given by the curve

$$\mathbf{r}(t) = \sin t\mathbf{i} - 5t\mathbf{j} - \cos t\mathbf{k}.$$

(a) (2pts) Find the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ of the particle at time t.

2

$$r'(t) = \cos t\,\hat{\imath} - 5\hat{\jmath} + \sin t\,\hat{k}$$

(b) (3pts) Find the arc length of the curve between times $t = 0$ and $t = 3$.

$$\|r'(t)\| = \sqrt{\cos^2 t + 25 + \sin^2 t} = \sqrt{26}$$

3

$$L = \int_0^3 \|r'(t)\|\,dt = \int_0^3 \sqrt{26}\,dt = 3\sqrt{26}$$

(c) (2pts) Find the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ of the particle at time t.

2

$$r''(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{k}$$

(d) (2pts) Find the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{||\mathbf{v}(t)||}$.

2

$$T(t) = \frac{1}{\sqrt{26}}(\cos t\,\hat{\imath} - 5\hat{\jmath} + \sin t\,\hat{k}) = \frac{\cos t}{\sqrt{26}}\hat{\imath} - \frac{5}{\sqrt{26}}\hat{\jmath} + \frac{\sin t}{\sqrt{26}}\hat{k}.$$

(e) (3pts) Find the principal unit normal vector $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$.

3

$$T'(t) = \frac{-\sin t}{\sqrt{26}}\hat{\imath} + \frac{\cos t}{\sqrt{26}}\hat{k}.$$

$$\Rightarrow N(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{k}$$

$$\|T'(t)\| = \frac{1}{\sqrt{26}}(\sin^2 t + \cos^2 t) = \frac{1}{\sqrt{26}}$$

(f) (5pts) Find the curvature $\kappa(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}$ of the particle's path at time t.

5

$$r'(t) \times r''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos t & -5 & \sin t \\ -\sin t & 0 & \cos t \end{vmatrix} = -5\cos t\,\hat{\imath} - \hat{\jmath} - 5\sin t\,\hat{k}$$

$$\Rightarrow \|r'(t) \times r''(t)\| = \sqrt{25+1} = \sqrt{26}$$

$$\kappa(t) = \frac{\sqrt{26}}{(\sqrt{26})^3} = \frac{1}{26}$$

or

$$\kappa(t) = \frac{\|T'(t)\|}{\|v(t)\|} = \frac{1/\sqrt{26}}{\sqrt{26}} = \frac{1}{26}.$$

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5. (12 pts) Match the equation with the type of surface it describes by writing the appropriate capital letter (**A-F**) in the provided blank. Each answer will be used exactly once.

2 each

(a) E $x^2 + y^2 - z^2 = 1$

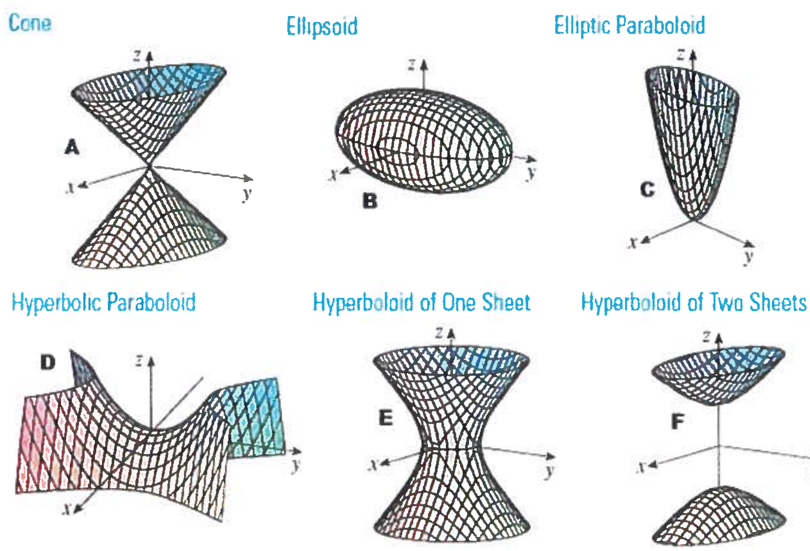
(b) B $3x^2 + y^2 + 3z^2 = 1$

(c) F $x^2 + y^2 - z^2 = -1$

(d) C $3x^2 + y^2 - z = 0$

(e) A $x^2 + 2y^2 - z^2 = 0$

(f) D $-x^2 + y^2 - z = 0$



(Images taken from page 808 of Stewart, *Calculus-Early Transcendentals*, 6e)

6. (10pts) Match the function with the description of its level sets ($z =$ constant) by writing the appropriate capital letter (**A-E**) in the provided blank. Each letter should be used exactly once.

2 each

B	$z = x^2 + y^2$	**A** a collection of parallel lines
E	$z = \sqrt{x^2 + 2y^2 - 1}$	**B** a collection of concentric circles
D	$z = \frac{y}{x}$	**C** a collection of hyperbolas
C	$z = x^2 - y^2$	**D** a collection of lines through the origin
A	$z = 3x - 2y$	**E** a collection of ellipses

7. (9pts) Convert between Cartesian, cylindrical, and spherical coordinates as indicated

1 each

(a) Find the Cartesian coordinates of the point with spherical coordinates $(\rho, \theta, \phi) = (\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{2})$

$x =$ -1 $y =$ 1 $z =$ 0

(b) Find the Cartesian coordinates of the point with cylindrical coordinates $(r, \theta, z) = (5, \frac{\pi}{6}, -2)$

$x =$ $5\sqrt{3}/2$ $y =$ $5/2$ $z =$ -2

(c) Find the cylindrical coordinates of the point with Cartesian coordinates $(x, y, z) = (3, -3, 1)$

$r =$ $3\sqrt{2}$ $\theta =$ $-\frac{\pi}{4}$ (or $\frac{7\pi}{4}$) $z =$ 1

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8. (12pts) Evaluate the following limits. Show your work. If they do not exist, write 'DNE' and explain why.

(a) $\lim_{(x,y)\to(0,0)} \frac{e^{x+y}+6}{1-x} = \frac{e^{0+0}+6}{1-0} = \frac{7}{1} = 7$

4

(b) $\lim_{(x,y)\to(0,0)} \frac{2x-3y}{x-y}$

Approach $(0,0)$ along x-axis $(y=0)$

$= \lim_{x\to 0} \frac{2x}{x} = 2.$

Approach $(0,0)$ along y-axis $(x=0)$

$= \lim_{y\to 0} \frac{-3y}{-y} = 3.$

So $\lim_{(x,y)\to(0,0)} \frac{2x-3y}{x-y}$ DNE

4

(c) $\lim_{(x,y)\to(0,0)} \frac{x^3-2y^4}{x^2+y^2}$ **Hint:** Use polar coordinates.

$= \lim_{r\to 0} \frac{r^3\cos^3\theta - 2r^4\sin^4\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta - 2r^2\sin^4\theta$

$= 0$ (since $\cos\theta$ and $\sin\theta$ are bounded between -1 and 1).

4

9. (12pts) Consider the function

$$f(x,y) = y^2\sin x + y^3 - x\cos y.$$

(a) (6pts) Find the equation of the tangent plane to the graph of $z = f(x,y)$ at the point $(0,\pi,\pi^3)$.

2 $f_x(x,y) = y^2\cos x - \cos y \Rightarrow f_x(0,\pi) = \pi^2\cos(0) - \cos(\pi) = \pi^2+1$

6 2 $f_y(x,y) = 2y\sin x + 3y^2 + x\sin y \Rightarrow f_y(0,\pi) = 2\pi\sin(0) + 3\pi^2 + 0\sin(\pi) = 3\pi^2$

2 $z = \pi^3 + (\pi^2+1)(x-0) + 3\pi^2(y-\pi)$

(or $z = (\pi^2+1)x + 3\pi^2 y - 2\pi^2$).

(b) (6pts) Find the following second derivatives:

2 i. $f_{xx}(x,y) =$ $-y^2\sin x$

6 2 ii. $f_{yy}(x,y) =$ $2\sin x + 6y + x\cos y$

2 iii. $f_{xy}(x,y) =$ $2y\cos x + \sin y$.

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