Name: 

**Instructions.** Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Page 5 is blank in case you need extra paper. Please circle your final answers.

1. (14pts) Consider the vectors \( \mathbf{u} = (-1, 5, 2) \) and \( \mathbf{v} = (2, 2, -3) \). Find
   
   (a) (2pts) \( 2 \mathbf{u} - 3 \mathbf{v} \)
   
   \[ 2 \langle -1, 5, 2 \rangle - 3 \langle 2, 2, -3 \rangle = \langle -2, 10, 4 \rangle - \langle 6, 6, -9 \rangle = \langle -8, 4, 13 \rangle \]
   
   (b) (2pts) \( ||\mathbf{u}|| \)
   
   \[ ||\mathbf{u}|| = \sqrt{(-1)^2 + 5^2 + 2^2} = \sqrt{30} \]
   
   (c) (2pts) \( \mathbf{u} \cdot \mathbf{v} \)
   
   \[ \mathbf{u} \cdot \mathbf{v} = (-1)(2) + (5)(2) + (2)(-3) = -2 + 10 - 6 = 2 \]
   
   (d) (4pts) the scalar projection of \( \mathbf{u} \) onto \( \mathbf{v} \). Recall, this is the dot product of \( \mathbf{u} \) with the unit vector pointing in the same direction as \( \mathbf{v} \).
   
   \[ \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||} = \frac{2}{\sqrt{30}} = \frac{2}{\sqrt{17}} \]
   
   (e) (4pts) \( \mathbf{u} \times \mathbf{v} \)
   
   \[ \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 5 & 2 \\ 2 & 2 & -3 \end{vmatrix} = \hat{i}(-15 - 4) + \hat{j}(10 - 3) + \hat{k}(2 - 10) \]
   
   \[ = -19\hat{i} + 7\hat{j} - 12\hat{k} = \langle -19, 7, -12 \rangle \]

2. (8pts) Find an equation of the plane containing the point \( (0, 5, -4) \) which is perpendicular to the line
   
   \[ r(t) = (1, -1, 4) + t(3, -2, -1) \]

   \[ r'(t) = \mathbf{n} = \langle 3, -2, -1 \rangle \]

   \[ 3x - 2y - z = c \]

   Contains the point \( (0, 5, -4) \)

   \[ c = 3(0) - 2(5) - (-4) = 0 - 10 + 4 = -6 \]

   \[ 3x - 2y - z = -6 \]
3. (6pts) Find the equation of the largest sphere centered at (2, 3, 5) that is completely contained in the first octant. Note: the first octant is where \( x \geq 0, y \geq 0, \) and \( z \geq 0. \)

Radius can be at most 2, since any larger radius would have two points with \( x < 0. \)

\[
(x - 2)^2 + (y - 3)^2 + (z - 5)^2 = 4.
\]

\[
(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.
\]

4. (17pts) Suppose a particle’s position at time \( t \) is given by the curve \( r(t) = \sin ti - 5tj - \cos tk. \)

(a) (2pts) Find the velocity \( v(t) = r'(t) \) of the particle at time \( t. \)

\[
r'(t) = \cos t \hat{i} - 5 \hat{j} + \sin t \hat{k}.
\]

(b) (3pts) Find the arc length of the curve between times \( t = 0 \) and \( t = 3. \)

\[
L = \int_{0}^{3} \sqrt{\|r'(t)\|^2} dt = \int_{0}^{3} \sqrt{26} dt = 3\sqrt{26}.
\]

(c) (2pts) Find the acceleration \( a(t) = r''(t) \) of the particle at time \( t. \)

\[
r''(t) = -\sin t \hat{i} + \cos t \hat{k}.
\]

(d) (2pts) Find the unit tangent vector \( T(t) = \frac{v(t)}{\|v(t)\|}. \)

\[
T(t) = \frac{1}{\sqrt{26}} \left( \cos t \hat{i} - 5 \hat{j} + \sin t \hat{k} \right) = \frac{\cos t}{\sqrt{26}} \hat{i} - \frac{5}{\sqrt{26}} \hat{j} + \frac{\sin t}{\sqrt{26}} \hat{k}.
\]

(e) (3pts) Find the principal unit normal vector \( N(t) = \frac{r'(t) \times r''(t)}{\|r'(t) \times r''(t)\|}. \)

\[
N(t) = -\sin t \hat{i} + \cos t \hat{k}.
\]

(f) (5pts) Find the curvature \( k(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} \) of the particle’s path at time \( t. \)

\[
\kappa(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{\sqrt{26}}{\|r'(t)\|^3} = \frac{1}{26}.
\]
5. (12 pts) Match the equation with the type of surface it describes by writing the appropriate capital letter (A-F) in the provided blank. Each answer will be used exactly once.

(a) $x^2 + y^2 - z^2 = 1$  
(b) $3x^2 + y^2 + 3z^2 = 1$  
(c) $x^2 + y^2 - z^2 = -1$  
(d) $3x^2 + y^2 - z = 0$  
(e) $x^2 + 2y^2 - z^2 = 0$  
(f) $-x^2 + y^2 - z = 0$

Cone Ellipsoid Elliptic Paraboloid

Hyperbolic Paraboloid Hyperboloid of One Sheet Hyperboloid of Two Sheets

6. (10pts) Match the function with the description of its level sets ($z = $ constant) by writing the appropriate capital letter (A-E) in the provided blank. Each letter should be used exactly once.

(a) $z = x^2 + y^2$  
(b) $z = \sqrt{x^2 + 2y^2 - 1}$  
(c) $z = \frac{y}{x}$  
(d) $z = x^2 - y^2$  
(e) $z = 3x - 2y$

A a collection of parallel lines  
B a collection of concentric circles  
C a collection of hyperbolas  
D a collection of lines through the origin  
E a collection of ellipses

7. (9pts) Convert between Cartesian, cylindrical, and spherical coordinates as indicated

(a) Find the Cartesian coordinates of the point with spherical coordinates $(\rho, \theta, \phi) = (\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{2})$

\[ x = -1 \quad y = 1 \quad z = 0 \]

(b) Find the Cartesian coordinates of the point with cylindrical coordinates $(r, \theta, z) = (5, \frac{\pi}{6}, -2)$

\[ x = \frac{5\sqrt{3}}{2} \quad y = \frac{5}{2} \quad z = -2 \]

(c) Find the cylindrical coordinates of the point with Cartesian coordinates $(x, y, z) = (3, -3, 1)$

\[ r = 3\sqrt{2} \quad \theta = \frac{-\pi}{4} \left( \sigma - \frac{7\pi}{4} \right) \quad z = 1 \]
8. (12pts) Evaluate the following limits. Show your work. If they do not exist, write ‘DNE’ and explain why.

(a) \( \lim_{(x,y) \to (0,0)} \frac{e^{x+y} + 6}{1 - x} = \frac{e^{0+0} + 6}{1 - 0} = \frac{7}{1} = 7 \) 

(b) \( \lim_{(x,y) \to (0,0)} \frac{2x - 3y}{x - y} \)

\[ \text{Approach } (0,0) \text{ along } x\text{-axis } (y=0) \]
\[ = \lim_{x \to 0} \frac{2x}{x} = 2 \]

\[ \text{Approach } (0,0) \text{ along } y\text{-axis } (x=0) \]
\[ = \lim_{y \to 0} \frac{-3y}{-y} = 3 \]

\[ \text{So } \lim_{(x,y) \to (0,0)} \frac{2x - 3y}{x - y} \text{ DNE} \]

(c) \( \lim_{(x,y) \to (0,0)} \frac{x^2 - 2y^4}{x^2 + y^2} \)

\[ \text{Hint: Use polar coordinates.} \]
\[ \lim_{r \to 0} \frac{r^2 \cos^2 \theta - 2r^4 \sin^4 \theta}{r^2} = \lim_{r \to 0} r \cos^2 \theta - 2r^2 \sin^4 \theta \]
\[ = 0 \text{ (since } \cos \theta \text{ and } \sin \theta \text{ are bounded between } -1 \text{ and } 1) \]

9. (12pts) Consider the function \( f(x,y) = y^2 \sin x + y^3 - x \cos y \).

(a) (6pts) Find the equation of the tangent plane to the graph of \( z = f(x,y) \) at the point \((0,\pi, \pi^3)\).

\[ f_x(x,y) = y^2 \cos x - \cos y \Rightarrow f_x(0,\pi) = \pi^2 \cos(0) - \cos(\pi) = \pi^2 + 1 \]

\[ f_y(x,y) = 2y \sin x + 3y^2 + x \sin y \Rightarrow f_y(0,\pi) = 2\pi \sin(0) + 3\pi^2 + 0 \sin(\pi) = 3\pi \]

\[ z = \pi^3 + (\pi^2 + 1)(x - 0) + 3\pi^2 (y - \pi) \]
\[ \Rightarrow z = (\pi^2 + 1)x + 3\pi^2 y - 2\pi^2 \]

(b) (6pts) Find the following second derivatives:

i. \( f_{xx}(x,y) = -y^2 \sin x \)

ii. \( f_{yy}(x,y) = 2 \sin x + 6y + x \cos y \)

iii. \( f_{xy}(x,y) = 2y \cos x + 3 \sin y \)