Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Page 5 is blank in case you need extra paper. Please circle your final answers.

1. (16pts) Consider the vectors \( \mathbf{u} = (6, 0, 2) \) and \( \mathbf{v} = (-1, 7, 3) \). Find

(a) (2pts) \( \mathbf{v} - 2\mathbf{u} \)

\[ \langle -1, 7, 3 \rangle - 2 \langle 6, 0, 2 \rangle = \langle -12, -7, -4 \rangle = \langle -13, 7, -1 \rangle \]

(b) (2pts) \( ||\mathbf{u}|| \)

\[ ||\mathbf{u}|| = \sqrt{6^2 + 0^2 + 2^2} = \sqrt{40} = 2\sqrt{10} \]

(c) (2pts) The unit vector which points in the same direction as \( \mathbf{u} \)

\[ \frac{\mathbf{u}}{||\mathbf{u}||} = \frac{1}{\sqrt{40}} \langle 6, 0, 2 \rangle = \langle \frac{6}{\sqrt{40}}, 0, \frac{2}{\sqrt{40}} \rangle = \langle \frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}} \rangle \]

(d) (2pts) \( \mathbf{u} \cdot \mathbf{v} \)

\[ \mathbf{u} \cdot \mathbf{v} = 6(-1) + 0(7) + 2(3) = 0 \]

(e) (1pt) Are \( \mathbf{u} \) and \( \mathbf{v} \) orthogonal? Circle one: \( \text{YES} \) \( \text{NO} \)

(f) (3pts) \( \mathbf{u} \times \mathbf{v} \)

\[ \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 2 \\ -1 & 7 & 3 \end{vmatrix} = \mathbf{i}(0 - 14) + \mathbf{j}(14 - 2) + \mathbf{k}(42 - 0) = \langle -14, -20, 42 \rangle \]

(g) (4pts) Two of the following quantities are zero (or the zero vector). Which ones? Circle two letters.

\[ \bigcirc \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) \quad \bigcirc \mathbf{u} \times \mathbf{u} \quad \bigcirc \mathbf{u} \cdot \mathbf{u} \quad \bigcirc \mathbf{v} \times (\mathbf{u} \times \mathbf{v}) \]

2. (7pts) Find an equation of the plane consisting of all points that are equidistant from the points \( P = (1, 0, -1) \) and \( Q = (3, 2, 1) \).

Midpoint = \( \langle \frac{1+3}{2}, \frac{0+2}{2}, \frac{-1+1}{2} \rangle = \langle 2, 1, 0 \rangle \)

\[ \mathbf{n} = \overrightarrow{PQ} = \langle 3-1, 2-0, 1-(-1) \rangle = \langle 2, 2, 2 \rangle. \]

So

\[ 2x + 2y + 2z = d \]

\[ 2(2) + 2(1) + 2(0) = d \]

\[ 2x + 2y + 2z = 6 \]
3. (17pts) Suppose a particle's position at time \( t \) is given by the curve

\[ r(t) = (\cos t + t \sin t)i + 4t^2j + (\sin t - t \cos t)k. \]

For this problem, it is helpful if you remember the trig identity \( \sin^2 t + \cos^2 t = 1 \).

(a) (2pts) Find the velocity \( \mathbf{v}(t) \) of the particle at time \( t \).

\[ \mathbf{v}(t) = r'(t) = \left< -\sin t + \sin t + t \cos t, 8t, \cos t - \cos t + t \sin t \right> = \left< -\sin t, 8t, t \sin t \right> = \left< \cos t, 8t, t \sin t \right> \]

(b) (3pts) Find the arc length of the curve between times \( t = 0 \) and \( t = 2 \).

\[ ||\mathbf{v}(t)|| = \sqrt{t^2 \cos^2 t + 64t^2 + t^2 \sin^2 t} = \sqrt{65t^2} = t \sqrt{65} \]

\[ L = \int_0^2 ||\mathbf{v}(t)||\,dt = \int_0^2 t \sqrt{65} \,dt = \left( \frac{\sqrt{65}t^2}{2} \right)_0^2 = 2\sqrt{65} \]

(c) (2pts) Find the acceleration \( \mathbf{a}(t) \) of the particle at time \( t \).

\[ \mathbf{a}(t) = r''(t) = \left< -\sin t - t \sin t, 8, \sin t + t \cos t \right> \]

(d) (2pts) Find the unit tangent vector \( \mathbf{T}(t) = \frac{\mathbf{v}(t)}{||\mathbf{v}(t)||} \).

\[ \mathbf{T}(t) = \frac{1}{t \sqrt{65}} \left< t \cos t, 8t, t \sin t \right> = \frac{1}{\sqrt{65}} \left< \cos t, 8, t \sin t \right> \]

(e) (3pts) Find the principal unit normal vector \( \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} \).

\[ \mathbf{T}'(t) = \sqrt{65} \left< -\sin t, 0, \cos t \right> = \frac{1}{\sqrt{65}} \sqrt{65 \sin^2 t + \cos^2 t} \]

\[ ||\mathbf{T}'(t)|| = \frac{1}{\sqrt{65}} \sqrt{65 \sin^2 t + \cos^2 t} = \frac{1}{\sqrt{65}} \left< -\sin t, 0, \cos t \right> \]

(f) (5pts) Find the curvature \( \kappa(t) \) of the particle's path at time \( t \).

\[ \kappa = \frac{||\mathbf{T}'(t)||}{||\mathbf{T}(t)||^3} = \frac{1}{t \sqrt{65}} \frac{1}{65t} = \frac{1}{65t^2} \]

4. (5pts) Suppose the acceleration of a particle is given by

\[ \mathbf{a}(t) = (2t, t + \sin t, e^{-t}) \]

If the particle's initial velocity is \( \mathbf{v}(0) = (2, -3, 1) \), what is the velocity of the particle at time \( t \)?

\[ \mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \left< t^2 + C_1 \frac{t^2}{2} - \cos t + C_2, -3t + C_3 \right> \]

\[ \left< 2t, 2, 1 \right> = \mathbf{v}(0) = \left< C_1 - 1 + C_2, -1 + C_3 \right> \Rightarrow \begin{align*} C_1 &= 2 \\ C_2 &= -2 \\ C_3 &= 2 \end{align*} \]

\[ \mathbf{v}(t) = \left< t^2 + 2t, \frac{t^2}{2} - \cos t - 2, -e^{-t} + 2 \right> \]

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5. (14pts) Match the equation with the type of surface it determines by writing the appropriate capital letter (A-G) in the provided blank. Each letter should be used exactly once.

- \( G \quad x^2 + y^2 + z^2 = 1 \)  \( \text{A Elliptic Paraboloid} \)
- \( C \quad x^2 + z^2 - y^2 = 1 \)  \( \text{B Ellipsoid} \)
- \( B \quad x^2 + 2y^2 + 3z^2 = 1 \)  \( \text{C Hyperboloid of one sheet} \)
- \( E \quad x^2 - 2y^2 - z = 0 \)  \( \text{D Hyperboloid of two sheets} \)
- \( F \quad y = x + 3z - 7 \)  \( \text{E Hyperbolic Paraboloid} \)
- \( A \quad x^2 + 2y^2 - z = 0 \)  \( \text{F Plane} \)
- \( D \quad z^2 = x^2 - y^2 = 1 \)  \( \text{G Sphere} \)

6. (8pts) Match the equation and the description of the surface by writing the appropriate capital letter (A-D) in the provided blank. Each letter should be used exactly once.

- (a) \( C \quad \) In cylindrical coordinates, the surface \( z = r^2 \).
- (b) \( A \quad \) In cylindrical coordinates, the surface \( r^2 + z^2 = 4 \).
- (c) \( D \quad \) In spherical coordinates, the surface \( \rho = 2 \cos \phi \).
- (d) \( B \quad \) In spherical coordinates, the surface \( \theta = \frac{3\pi}{4} \).

A  a sphere centered at the origin.
B  a half-plane.
C  a paraboloid
D  a sphere centered at the point \((0, 0, 1)\) in Cartesian coordinates.

7. (9pts) Convert between Cartesian, cylindrical, and spherical coordinates as indicated. Please simplify as much as possible.

- (a) Find the cylindrical coordinates of the point with Cartesian coordinates \((-1, 1, 3)\)

  \( r = \sqrt{2} \quad \theta = \frac{3\pi}{4} \quad z = 3 \)

- (b) Find the spherical coordinates of the point with Cartesian coordinates \((1, \sqrt{3}, -2)\)

  \( \rho = \sqrt{8} \quad \theta = \frac{\pi}{3} \quad \phi = \frac{3\pi}{4} \)

- (c) Find the Cartesian coordinates of the point with cylindrical coordinates \((2, \frac{\pi}{6}, \pi)\)

  \( x = \sqrt{3} \quad y = 1 \quad z = \pi \)
8. (12pts) Evaluate the following limits. If they do not exist, write ‘DNE’ and explain why.

(a) \( \lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}}{1+x^2+y^2} = \frac{e^{0+0}}{1+0+0} = \frac{1}{1} = 1 \)

Since denominator is defined at \((x,y)=(0,0)\).

(b) \( \lim_{(x,y)\to(0,0)} \frac{x^2+y}{x^2-y} \) \( \text{DNE} \)

If we approach \((0,0)\) along \(x\)-axis \((y=0)\), then limit equals
\( \lim_{x\to0} \frac{x^2}{x^2} = 1. \)

If we approach \((0,0)\) along \(y\)-axis \((x=0)\), then limit equals
\( \lim_{y\to0} \frac{y}{-y} = -1. \)

(c) \( \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^2+y^2} \) Hint: Use polar coordinates.
\( = \lim_{r\to0} \frac{r^3\cos^3\theta \sin\theta}{r^2} = \lim_{r\to0} r^2 (\cos^3\theta \sin\theta) = 0 \)

independent of \(\theta\).

(d) \( \lim_{h\to0} \frac{\sin((x+h)y) - \sin(xy)}{h} \) \( \text{Hint: Think derivative.} \)
\( = \frac{d}{dx} (\sin(xy)) = y \cos(xy) \)

9. (12pts) Consider the function \( f(x, y) = xy \cos(x^2) \).

Compute the following partial derivatives:

(a) \( f_x(x, y) = y \cos(x^2) - 2x^2y \sin(x^2) \)

(b) \( f_y(x, y) = x \cos(x^2) \)

(c) \( f_{yy}(x, y) = 0 \)

(d) Find \( \nabla f(0, 2) \). That is find the gradient of \( f \) at the point \((0, 2)\).

\( \nabla f(0, 2) = (2 \cos(0) - 0, 0 \cdot \cos(0)) \)
\( = (2, 0) \)