2210-90 Exam 1

Summer 2013

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Page 5 is blank in case you need extra paper. Please circle your final answers.

1. (16pts) Consider the vectors $\mathbf{u} = \langle 6, 0, 2\rangle$ and $\mathbf{v} = \langle -1, 7, 3\rangle$. Find

   (a) (2pts) $\mathbf{v} - 2\mathbf{u}$

   2

   $\langle -1,7,3\rangle - 2\langle 6,0,2\rangle = \langle -1-12, 7-0, 3-4\rangle = \langle -13, 7, -1\rangle$

   (b) (2pts) $||\mathbf{u}||$

   2

   $\|\bar{u}\| = \sqrt{6^2+0^2+2^2} = \sqrt{40} = 2\sqrt{10}$

   (c) (2pts) The unit vector which points in the same direction as $\mathbf{u}$

   2

   $\dfrac{\bar{u}}{\|\bar{u}\|} = \dfrac{1}{\sqrt{40}}\langle 6,0,2\rangle = \langle 6/\sqrt{40}, 0, 2/\sqrt{40}\rangle = \langle 3/\sqrt{10}, 0, 1/\sqrt{10}\rangle$

   (d) (2pts) $\mathbf{u}\cdot\mathbf{v}$

   2

   $\bar{u}\cdot\bar{v} = 6(-1) + 0(7) + 2(3) = 0$

   1

   (e) (1pt) Are $\mathbf{u}$ and $\mathbf{v}$ orthogonal? Circle one: YES NO

   (f) (3pts) $\mathbf{u}\times\mathbf{v}$

   3

   $$\bar{u}\times\bar{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 0 & 2 \\ -1 & 7 & 3 \end{vmatrix} = \hat{\imath}(-14) + \hat{\jmath}(-20) + \hat{k}(42) = \langle -14, -20, 42\rangle$$

   (g) (4pts) Two of the following quantities are zero (or the zero vector). Which ones? Circle two letters.

   4

   A. $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})$ B. $\mathbf{u}\times\mathbf{u}$ C. $\mathbf{u}\cdot\mathbf{u}$ D. $\mathbf{v}\times(\mathbf{u}\times\mathbf{v})$

2. (7pts) Find an equation of the plane consisting of all points that are equidistant from the points $P = (1, 0, -1)$ and $Q = (3, 2, 1)$.

midpoint $= \frac{1}{2}(1+3, 0+2, -1+1) = (2,1,0)$

$\bar{n} = \overrightarrow{PQ} = \langle 3-1, 2-0, 1-(-1)\rangle = \langle 2,2,2\rangle$.

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So

$2x+2y+2z = d$

$2(2)+2(1)+2(0) = 6$

$2x+2y+2z = 6$

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3. (17pts) Suppose a particle's position at time t is given by the curve

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + 4t^2\mathbf{j} + (\sin t - t\cos t)\mathbf{k}.$$

For this problem, it is helpful if you remember the trig identity $\sin^2 t + \cos^2 t = 1$.

(a) (2pts) Find the velocity $\mathbf{v}(t)$ of the particle at time t.

2

$$v(t) = r'(t) = \langle -\sin t + \sin t + t\cos t,\ 8t,\ \cos t - \cos t + t\sin t\rangle$$
$$= \langle t\cos t, 8t, t\sin t\rangle$$

(b) (3pts) Find the arc length of the curve between times $t = 0$ and $t = 2$.

$$\|v(t)\| = \sqrt{t^2\cos^2 t + 64t^2 + t^2\sin^2 t} = \sqrt{65t^2} = t\sqrt{65}.$$

3

$$L = \int_0^2 \|v(t)\|\,dt = \int_0^2 t\sqrt{65}\,dt = \left(\sqrt{65}\frac{t^2}{2}\right|_0^2 = 2\sqrt{65}$$

(c) (2pts) Find the acceleration $\mathbf{a}(t)$ of the particle at time t.

2

$$a(t) = r''(t) = \langle \cos t - t\sin t,\ 8,\ \sin t + t\cos t\rangle$$

(d) (2pts) Find the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{||\mathbf{v}(t)||}$.

2

$$T(t) = \frac{1}{t\sqrt{65}}\langle t\cos t, 8t, t\sin t\rangle = \frac{1}{\sqrt{65}}\langle \cos t, 8, \sin t\rangle$$

(e) (3pts) Find the principal unit normal vector $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$.

3

$$T'(t) = \frac{1}{\sqrt{65}}\langle -\sin t, 0, \cos t\rangle \qquad N(t) = \langle -\sin t, 0, \cos t\rangle$$
$$\|T'(t)\| = \frac{1}{\sqrt{65}}\sqrt{\sin^2 t + \cos^2 t} = \frac{1}{\sqrt{65}}$$

(f) (5pts) Find the curvature $\kappa(t)$ of the particle's path at time t.

$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{1/\sqrt{65}}{t\sqrt{65}} = \frac{1}{65t}$$

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<u>or</u>

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\|\langle 8t^2\cos t, -t^2, 8t^2\sin t\rangle\|}{(t\sqrt{65})^3} = \frac{t^2\sqrt{65}}{t^3(\sqrt{65})^3} = \frac{1}{65t}.$$

4. (5pts) Suppose the acceleration of a particle is given by

$$\mathbf{a}(t) = \langle 2t, t + \sin t, e^{-t}\rangle.$$

If the particle's initial velocity is $\mathbf{v}(0) = \langle 2, -3, 1\rangle$, what is the velocity of the particle at time t?

5

$$v(t) = \int a(t)\,dt = \langle t^2 + c_1, \frac{t^2}{2} - \cos t + c_2, -e^{-t} + c_3\rangle$$

$$\langle 2, -3, 1\rangle = v(0) = \langle c_1, -1 + c_2, -1 + c_3\rangle \Rightarrow \begin{array}{l} c_1 = 2 \\ c_2 = -2 \\ c_3 = 2 \end{array}$$

$$v(t) = \langle t^2 + 2, \frac{t^2}{2} - \cos t - 2, -e^{-t} + 2\rangle$$

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5. (14pts) Match the equation with the type of surface it determines by writing the appropriate capital letter (**A-G**) in the provided blank. Each letter should be used exactly once.

G	$x^2 + y^2 + z^2 = 1$	**A** Elliptic Paraboloid
C	$x^2 + z^2 - y^2 = 1$	**B** Ellipsoid
B	$x^2 + 2y^2 + 3z^2 = 1$	**C** Hyperboloid of one sheet
E	$x^2 - 2y^2 - z = 0$	**D** Hyperboloid of two sheets
F	$y = x + 3z - 7$	**E** Hyperbolic Paraboloid
A	$x^2 + 2y^2 - z = 0$	**F** Plane
D	$z^2 - x^2 - y^2 = 1$	**G** Sphere

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6. (8pts) Match the equation and the description of the surface by writing the appropriate capital letter (**A-D**) in the provided blank. Each letter should be used exactly once.

(a) C In cylindrical coordinates, the surface $z = r^2$.

(b) A In cylindrical coordinates, the surface $r^2 + z^2 = 4$.

(c) D In spherical coordinates, the surface $\rho = 2\cos\phi$.

(d) B In spherical coordinates, the surface $\theta = \frac{3\pi}{4}$.

A a sphere centered at the origin.

B a half-plane.

C a paraboloid

D a sphere centered at the point $(0, 0, 1)$ in Cartesian coordinates.

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7. (9pts) Convert between Cartesian, cylindrical, and spherical coordinates as indicated. Please simplify as much as possible.

(a) Find the cylindrical coordinates of the point with Cartesian coordinates $(-1, 1, 3)$

$r =$ $\sqrt{2}$ $\quad \theta =$ $3\pi/4$ $\quad z =$ 3

(b) Find the spherical coordinates of the point with Cartesian coordinates $(1, \sqrt{3}, -2)$

$\rho =$ $\sqrt{8}$ $\quad \theta =$ $\pi/3$ $\quad \phi =$ $3\pi/4$

(c) Find the Cartesian coordinates of the point with cylindrical coordinates $(2, \frac{\pi}{6}, \pi)$

$x =$ $\sqrt{3}$ $\quad y =$ 1 $\quad z =$ π

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8. (12pts) Evaluate the following limits. If they do not exist, write 'DNE' and explain why.

(a) $\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}}{1+x^2+y^2} = \frac{e^{0+0}}{1+0+0} = \frac{1}{1} = 1$

3

since denominator is defined at $(x,y)=(0,0)$.

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2+y}{x^2-y}$ DNE

3

If we approach $(0,0)$ along x-axis $(y=0)$, then limit equals $\lim_{x\to0} \frac{x^2}{x^2} = 1$.

If we approach $(0,0)$ along y-axis $(x=0)$, then limit equals $\lim_{y\to0} \frac{y}{-y} = -1$.

(c) $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^2+y^2}$ **Hint:** Use polar coordinates.

3

$= \lim_{r\to0} \frac{r^3\cos^3\theta\, r\sin\theta}{r^2} = \lim_{r\to0} r^2(\cos^3\theta \sin\theta) = 0$

independent of θ.

(d) $\lim_{h\to0} \frac{\sin((x+h)y) - \sin(xy)}{h}$ **Hint:** Think derivative.

3

$= \frac{\partial}{\partial x}(\sin(xy)) = y\cos(xy)$

9. (12pts) Consider the function

$$f(x,y) = xy\cos(x^2).$$

Compute the following partial derivatives:

(a) $f_x(x,y) =$

3

$y\cos(x^2) - 2x^2y\sin(x^2)$

(b) $f_y(x,y) =$

3

$x\cos(x^2)$

(c) $f_{yy}(x,y) =$

3

0

(d) Find $\nabla f(0,2)$. That is find the gradient of f at the point $(0,2)$.

3

$\nabla f(0,2) = \langle 2\cos(0) - 0,\ 0\cdot\cos(0)\rangle$

$= \langle 2, 0\rangle$

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