2210-90 Exam 1

Spring 2014

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Page 5 is blank in case you need extra paper. Please circle your final answers.

1. (15pts) Consider the vectors $\mathbf{u} = \langle 1, -1, 5 \rangle$ and $\mathbf{v} = \langle 2, -2, 3 \rangle$. Find

   (a) (2pts) $5\mathbf{u}$

   2 $= \langle 5, -5, 25 \rangle$

   (b) (2pts) $\mathbf{u} - \mathbf{v}$

   2 $= \langle 1, -1, 5 \rangle - \langle 2, -2, 3 \rangle = \langle -1, 1, 2 \rangle$

   (c) (2pts) $||\mathbf{u}||$

   2 $= \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27}$

   (d) (2pts) The unit vector which points in the same direction as $\mathbf{u}$

   2 $= \frac{1}{||u||} \bar{u} = \frac{1}{\sqrt{27}} \langle 1, -1, 5 \rangle$

   (e) (2pts) $\mathbf{u} \cdot \mathbf{v}$

   2 $= (1)(2) + (-1)(-2) + (5)(3) = 2 + 2 + 15 = 19$

   (f) (2pts) Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$. Answer in the form $\theta = \cos^{-1}(x)$.

   2 $\bar{u} \cdot \bar{v} = ||\bar{u}|| \, ||\bar{v}|| \cos\theta \Rightarrow 19 = \sqrt{27} \cdot \sqrt{17} \cos\theta$

   $||\bar{v}|| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{17}$ $\qquad \theta = \cos^{-1}\left(19 / \sqrt{27} \cdot \sqrt{17} \right)$

   (g) (3pts) $\mathbf{u} \times \mathbf{v}$

   3 $$\bar{u} \times \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 5 \\ 2 & -2 & 3 \end{vmatrix} = \hat{i}(-3+10) - \hat{j}(3-10) + \hat{k}(-2+2)$$

   $= 7\hat{i} + 7\hat{j}$

2. (5pts) Find an equation of the plane containing the point $(1, -1, 3)$ which **never** intersects the plane

$$x - 2y + 5z = 9$$

5 If they never intersect, the planes must have the same orientation, i.e. normal vector.

$x - 2y + 5z = d$

passes through $(1, -1, 3)$

$d = 1 - 2(-1) + 5(3) = 1 + 2 + 15 = 18. \Rightarrow x - 2y + 5z = 18$

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3. (6pts) The line segment connecting $(0, 2, -4)$ to $(2, 2, 6)$ is a diameter of a sphere. Find an equation of this sphere.

midpoint $= (1, 2, 1)$

length of line segment $=$ diameter of sphere $= \sqrt{2^2+0^2+10^2} = \sqrt{104}$

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radius of sphere $= \frac{\sqrt{104}}{2} = \sqrt{26}$

$$(x-1)^2 + (y-2)^2 + (z-1)^2 = 26$$

4. (12pts) Suppose a particle's position at time t is given by the curve

$$\mathbf{r}(t) = 6t\mathbf{i} + 9\mathbf{j} + (t^2+1)\mathbf{k}.$$

(a) (2pts) Find the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ of the particle at time t.

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$$r'(t) = 6\hat{i} + 0\hat{j} + 2t\hat{k}$$

(b) (3pts) Set up an integral which gives the arc length of the curve between times $t = 0$ and $t = 7$. **Do not evaluate.**

$$\|r'(t)\| = \sqrt{6^2 + (2t)^2} = \sqrt{36+4t^2}$$

3

$$L = \int_0^7 \|r'(t)\| \, dt = \int_0^7 \sqrt{36+4t^2} \, dt$$

(c) (2pts) Find the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ of the particle at time t.

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$$r''(t) = 2\hat{k}$$

(d) (5pts) Find the curvature $\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$ of the particle's path at time t.

$$\kappa(t) = \frac{12}{(36+4t^2)^{3/2}}$$

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$$r'(t) \times r''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 0 & 2t \\ 0 & 0 & 2 \end{vmatrix} = -12\hat{j}$$

5. (6pts) Suppose the velocity of a particle is given by

$$\mathbf{v}(t) = \langle \sin t, e^t, 3\cos t \rangle.$$

If the particle's initial position is $\mathbf{r}(0) = \langle 1, 1, 3 \rangle$, where is the particle at time $t = \pi$?

$$v(t) = r'(t) = \langle \sin t, et, 3\cos t \rangle$$

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$$r(t) = \left\langle \int \sin t \, dt, \int e^t dt, \int 3\cos t \, dt \right\rangle = \langle -\cos t + C_1, e^t + C_2, 3\sin t + C_3 \rangle$$

$$\langle 1, 1, 3 \rangle = r(0) = \langle -1 + C_1, 1 + C_2, C_3 \rangle$$ so

$C_1 = 2$, $C_2 = 0$, $C_3 = 3$.

$$\boxed{r(t) = \langle 2 - \cos t, e^t, 3\sin t + 3 \rangle} \quad r(\pi) = \langle 3, e^\pi, 3 \rangle$$

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6. (14pts) Match the equation with the type of surface it determines by writing the appropriate capital letter (**A-G**) in the provided blank. Each letter should be used exactly once.

F	$z = 9x - y + 7$	**A** Elliptic Paraboloid
A	$y = x^2 + 5z^2$	**B** Ellipsoid
B	$2x^2 + y^2 + 9z^2 = 3$	**C** Hyperboloid of one sheet
E	$y^2 - x^2 - z = 1$	**D** Hyperboloid of two sheets
G	$x^2 + y^2 + z^2 = 3$	**E** Hyperbolic Paraboloid
C	$x^2 + 2y^2 - z^2 = 1$	**F** Plane
D	$x^2 - 2y^2 - z^2 = 1$	**G** Sphere

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7. (8pts) Match the equation and the description of the surface by writing the appropriate capital letter (**A-D**) in the provided blank. Each letter should be used exactly once.

(a) C In cylindrical coordinates, the surface $r = 4$.
(b) D In cylindrical coordinates, the surface $r^2 - z^2 = 4$.
(c) B In spherical coordinates, the surface $\phi = \frac{\pi}{2}$.
(d) A In spherical coordinates, the surface $\rho = 4$.

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A a sphere.
B a plane.
C a cylinder.
D a hyperboloid.

8. (9pts) Convert between Cartesian, cylindrical, and spherical coordinates as indicated. Please simplify as much as possible.

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(a) Find the cylindrical coordinates of the point with Cartesian coordinates $(-1, -1, 5)$

$r =$ $\sqrt{2}$ $= \sqrt{1^2+1^2}$ $\theta =$ $5\pi/4$ not $\tan^{-1}(1) = \pi/4$ $z =$ 5

(b) Find the spherical coordinates of the point with Cartesian coordinates $(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$

$\rho =$ 4 $= \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2 + (2\sqrt{3})^2}$ $\theta =$ $\pi/4$ $= \tan^{-1}(1)$ $\phi =$ $\pi/6$ $= \cos^{-1}\left(2\sqrt{3}/4\right) = \cos^{-1}\left(\sqrt{3}/2\right)$

(c) Find the Cartesian coordinates of the point with cylindrical coordinates $(2, -\frac{\pi}{3}, \sqrt{3})$

$x =$ 1 $= 2\cos\left(-\frac{\pi}{3}\right)$ $y =$ $-\sqrt{3}$ $= 2\sin\left(-\frac{\pi}{3}\right)$ $z =$ $\sqrt{3}$

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9. (12pts) Evaluate the following limits. If they do not exist, write 'DNE' and explain why.

(a) $\lim_{(x,y)\to(0,0)} \ln(x^2+y^2+1)$ $= \ln(1) = 0$

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(b) $\lim_{(x,y)\to(0,0)} \frac{(x+2y)^2}{x^2+4y^2}$

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Approach $(0,0)$ along x-axis $(y=0)$
$= \lim_{x\to 0} \frac{x^2}{x^2} = 1.$

Approach $(0,0)$ along $x=y$
$= \lim_{x\to 0} \frac{(3x)^2}{5x^2} = \frac{9}{5}.$

So $\lim_{(x,y)\to(0,0)} \frac{(x+2y)^2}{x^2+4y^2}$ DNE

(c) $\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2}$ **Hint:** Use polar coordinates.

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$= \lim_{r\to 0} \frac{r^3 \sin^3\theta}{r^2} = \lim_{r\to 0} r\sin^3\theta = \sin^3\theta\left(\lim_{r\to 0} r\right) = 0$ (indep. of θ)

10. (13pts) Consider the function

$$f(x,y) = xe^{-3y} + x^2y^2.$$

(a) (7pts) Find the equation of the tangent plane to the graph of $z = f(x,y)$ at the point $(1,0,1)$.

2 $f_x(x,y) = e^{-3y} + 2xy^2 \Rightarrow f_x(1,0) = 1.$

2 $f_y(x,y) = -3xe^{-3y} + 2x^2y \Rightarrow f_y(1,0) = -3.$

$f(1,0) = 1.$

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

3 $z = 1 + 1(x-1) - 3(y-0)$
$= x - 3y$

(b) (6pts) Find the following second derivatives:

2 i. $f_{xx}(x,y) =$ $2y^2$

2 ii. $f_{yy}(x,y) =$ $9xe^{-3y} + 2x^2$

2 iii. $f_{xy}(x,y) =$ $-3e^{-3y} + 4xy$

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