2210-90 Exam 1

Spring 2013

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Page 5 is blank in case you need extra paper. Please circle your final answers.

1. (16pts) Consider the vectors $\mathbf{u} = \langle 0, 2, -1 \rangle$ and $\mathbf{v} = \langle 3, 2, 3 \rangle$. Find

   (a) (2pts) $\mathbf{u} - 3\mathbf{v}$

   2 $\langle 0,2,-1\rangle - 3\langle 3,2,3\rangle = \langle -9,-4,-10\rangle$

   (b) (2pts) $||\mathbf{u}||$

   2 $\|\bar{u}\| = \sqrt{0^2+2^2+1^2} = \sqrt{5}$

   (c) (2pts) $\mathbf{u} \cdot \mathbf{v}$

   2 $\langle 0,2,-1\rangle \cdot \langle 3,2,3\rangle = (0)(3)+(2)(2)+(-1)(3) = 0+4-3 = 1$

   (d) (2pts) Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$. Answer in the form $\theta = \cos^{-1}(x)$.

   $\bar{u}\cdot\bar{v} = \|\bar{u}\|\|\bar{v}\|\cos\theta$ $\qquad \|\bar{v}\| = \sqrt{3^2+2^2+3^2} = \sqrt{22}$

   2 $1 = \sqrt{5}\cdot\sqrt{22}\cos\theta \Rightarrow \theta = \cos^{-1}\left(1/\sqrt{110}\right)$

   (e) (4pts) $\mathbf{u} \times \mathbf{v}$

   4 $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -1 \\ 3 & 2 & 3 \end{vmatrix} = \hat{i}(6+2) - \hat{j}(0+3) + \hat{k}(0-6) = \langle 8,-3,-6\rangle$

   (f) (4pts) Find the vector projection of $\mathbf{u}$ onto $\mathbf{v}$

   4 $\dfrac{\bar{u}\cdot\bar{v}}{\|v\|^2}\bar{v} = \dfrac{1}{22}\langle 3,2,3\rangle = \left\langle \dfrac{3}{22}, \dfrac{2}{22}, \dfrac{3}{22}\right\rangle$

2. (8pts) Find a unit vector which is perpendicular to both of the vectors $\mathbf{a} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

8 $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 5 & -2 & 1 \end{vmatrix} = \hat{i}(1-2) - \hat{j}(2+5) + \hat{k}(-4-5)$ trying cross product +3

4 $= -\hat{i} - 7\hat{j} - 9\hat{k}$

2 $\|\bar{a}\times\bar{b}\| = \sqrt{1^2+7^2+9^2} = \sqrt{1+49+81} = \sqrt{131}$

$\bar{u} = \dfrac{-1}{\sqrt{131}}\hat{i} - \dfrac{7}{\sqrt{131}}\hat{j} - \dfrac{9}{\sqrt{131}}\hat{k}$ plus or minus

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3. (12pts) Consider the points $P(1,3,-1)$ and $Q(2,2,5)$.

(a) (3pts) Find the vector $\overrightarrow{PQ}$ that points in the direction from P to Q; that is find the vector with a tail at P and head at Q.

3

$$\overrightarrow{PQ} = \langle 2,2,5\rangle - \langle 1,3,-1\rangle = \langle 1,-1,6\rangle$$

(b) (3pts) Find the midpoint of P and Q.

3

$$m = \frac{1}{2}\left(\langle 2,2,5\rangle + \langle 1,3,-1\rangle\right) = \frac{1}{2}\langle 3,5,4\rangle = \left(\frac{3}{2},\frac{5}{2},2\right)$$

(c) (6pts) Find the equation of the plane which is equidistant from P and Q. **Hint:** This plane has normal vector $\overrightarrow{PQ}$ and contains the midpoint.

6

$\bar{n} = \langle 1,-1,6\rangle$

$x - y + 6z = D$

contains $(3/2, 5/2, 2)$ so

$\frac{3}{2} - \frac{5}{2} + 6(2) = D$

$D = 11.$

$$x - y + 6z = 11$$

4. (8pts) The set of all points (x,y,z) which satisfy

$$x^2 + y^2 + z^2 - 4x + 6y - 16z = 0$$

determines a sphere. Determine the center point (h,k,l) and the radius r of the sphere. Write your final answers in the provided blanks. **Hint:** Complete the squares.

8

$h =$ 2

$k =$ -3

$l =$ 8

$r =$ $\sqrt{77}$

$x^2+y^2+z^2-4x+6y-16z=0$

$(x^2-4x)+(y^2+6y)+(z^2-16z)=0$

$(x-2)^2-4+(y+3)^2-9+(z-8)^2-64=0$

$(x-2)^2+(y+3)^2+(z-8)^2=77$

5. (20pts) Suppose a particle's position at time t is given by the curve

$$\mathbf{r}(t) = 2\cos(2t)\mathbf{i} + 2\sin(2t)\mathbf{j} + 3t\mathbf{k}.$$

(a) (2pts) Find the velocity $\mathbf{v}(t)$ of the particle at time t.

2 $\bar{v}(t) = \bar{r}'(t) = -4\sin(2t)\hat{i} + 4\cos(2t)\hat{j} + 3\hat{k}$

(b) (3pts) Find the arc length of the curve between times $t = 0$ and $t = 3$.

$\|v(t)\| = \sqrt{16\sin^2(2t) + 16\cos^2(2t) + 9} = \sqrt{25} = 5$

3 $L = \int_0^3 \|v(t)\|\,dt = \int_0^3 5\,dt = 15$

(c) (2pts) Find the acceleration $\mathbf{a}(t)$ of the particle at time t.

$= \bar{v}'(t) = \cancel{r''(t)}$

2 $\bar{a}(t) = -8\cos(2t)\hat{i} - 8\sin(2t)\hat{j}$

(d) (2pts) Find the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$.

2 $\bar{T}(t) = \frac{1}{5}\bar{v}(t) = -\frac{4}{5}\sin(2t)\hat{i} + \frac{4}{5}\cos(2t)\hat{j} + \frac{3}{5}\hat{k}$

(e) (3pts) Find the principal unit normal vector $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.

$\bar{T}'(t) = -\frac{8}{5}\cos(2t)\hat{i} - \frac{8}{5}\sin(2t)\hat{j}$

3 $\|\bar{T}'(t)\| = 8/5$ $\quad \bar{N}(t) = -\cos(2t)\hat{i} - \sin(2t)\hat{j}$

(f) (6pts) Find the curvature $\kappa(t)$ of the particle's path ~~at time~~ t.

6 $\kappa = \frac{\|r' \times r''\|}{\|r'\|^3}$ $\left(\text{or } \kappa = \frac{\|T'(t)\|}{\|v(t)\|} \right)$ $\quad \kappa = \frac{\sqrt{24^2 + 32^2}}{125} = \frac{8}{25}$

$$\bar{r}' \times \bar{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4\sin(2t) & 4\cos(2t) & 3 \\ -8\cos(2t) & -8\sin(2t) & 0 \end{vmatrix} = (-24\sin(2t))\hat{i} - 24\cos(2t)\hat{j} + 32\hat{k}$$

$\|r' \times r''\| = \sqrt{24^2 + 32^2}$

(g) (2pts) For all time t, this curve is contained on what type of surface? Circle the correct letter

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- A a sphere of radius 3 centered at the origin
- B a hyperboloid of one sheet parallel to the y-axis
- C a cylinder of radius 2 parallel to the z-axis

6. (16pts) Match the equation with its graph on the right by writing the appropriate capital letter (**A-H**) in the provided blank.

2 pts each

(a) D $2x^2 + 2y^2 + z^2 = 1$

(b) C $x^2 + z^2 - y^2 = -1$

(c) A $y^2 = x^2 + z^2$

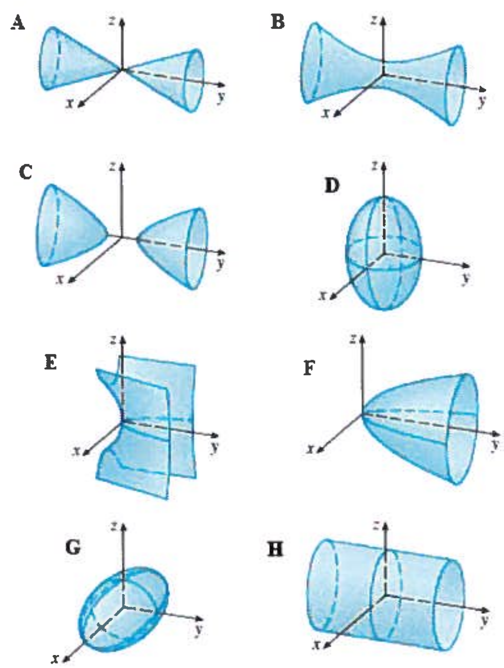
(d) H $x^2 + z^2 = 1$

(e) F $y = x^2 + z^2$

(f) G $x^2 + 2y^2 + 2z^2 = 1$

(g) B $x^2 + z^2 - y^2 = 1$

(h) E $y = x^2 - z^2$



7. (8pts) Match the equation and the description of the surface by writing the appropriate capital letter (**A-D**) in the provided blank. Each letter should be used exactly once.

2 pts each

(a) B In cylindrical coordinates, the surface $r = 5$.

(b) C In cylindrical coordinates, the surface $z = r$.

(c) A In spherical coordinates, the surface $\rho = 1$.

(d) D In spherical coordinates, the surface $\phi = \frac{3\pi}{4}$.

A sphere centered at the origin.

B cylinder parallel to the z-axis.

C cone opening in the positive z direction.

D cone opening in the negative z direction.

8. (12pts) Convert between Cartesian, cylindrical, and spherical coordinates as indicated.

1 pt each

(a) Find the cylindrical coordinates of the point with Cartesian coordinates $(\sqrt{3}, 1, -2)$

$r =$ 2 $\theta =$ $\pi/6$ $= \tan^{-1}(1/\sqrt{3})$ $z =$ -2

(b) Find the spherical coordinates of the point with Cartesian coordinates $(1, 1, \sqrt{2})$

$\rho =$ 2 $\theta =$ $\pi/4$ $= \tan^{-1}(1)$ $\phi =$ $\pi/4$ $= \cos^{-1}(\sqrt{2}/2)$

(c) Find the Cartesian coordinates of the point with spherical coordinates $(2, \frac{\pi}{6}, \frac{\pi}{2})$

$x =$ $\sqrt{3}$ $= 2\sin(\frac{\pi}{2})\cos(\frac{\pi}{6})$ $y =$ 1 $= 2\sin(\frac{\pi}{2})\sin(\frac{\pi}{6})$ $z =$ 0 $= 2\cos(\pi/2)$

(d) Find the Cartesian coordinates of the point with cylindrical coordinates $(4\sqrt{2}, -\frac{\pi}{4}, \pi)$

$x =$ 4 $= 4\sqrt{2}\cos(\frac{-\pi}{4})$ $y =$ -4 $= 4\sqrt{2}\sin(\frac{-\pi}{4})$ $z =$ π

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