2210-90 Exam 1

Fall 2013

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Please circle your final answers.

1. (10pts) Consider the vectors $\mathbf{u} = \langle 1, 5, -1 \rangle$ and $\mathbf{v} = \langle 4, 1, 1 \rangle$. Find

   (a) (2pts) $\mathbf{v} + 3\mathbf{u}$

   2 $\langle 4,1,1 \rangle + 3\langle 1,5,-1 \rangle = \langle 7,16,-2 \rangle$

   (b) (2pts) $||\mathbf{u}||$

   2 $\|\bar{u}\| = \sqrt{1^2+5^2+1^2} = \sqrt{27}$

   (c) (3pts) $\mathbf{u} \cdot \mathbf{v}$

   3 $\bar{u} \cdot \bar{v} = (1)(4) + (5)(1) + (-1)(1) = 8$

   (d) (3pts) $\mathbf{u} \times \mathbf{v}$

   3 $$\bar{u} \times \bar{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 5 & -1 \\ 4 & 1 & 1 \end{vmatrix} = 6\hat{\imath} - 5\hat{\jmath} - 19\hat{k} = \langle 6,-5,-19 \rangle$$

2. (7pts) Find a unit vector that is orthogonal to the plane containing the points $P = (1, 1, -1)$, $Q = (2, 0, 1)$, and $R = (0, 0, 5)$.

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   $\overrightarrow{PQ} = \langle 2,0,1 \rangle - \langle 1,1,-1 \rangle = \langle 1,-1,2 \rangle$

   $\overrightarrow{PR} = \langle 0,0,5 \rangle - \langle 1,1,-1 \rangle = \langle -1,-1,6 \rangle$

   $$\overrightarrow{PQ} \times \overrightarrow{PR} = \bar{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ -1 & -1 & 6 \end{vmatrix} = -4\hat{\imath} - 8\hat{\jmath} - 2\hat{k}$$

   orthogonal, but not <u>unit</u> vector

   $$\bar{u} = \frac{1}{\|\bar{n}\|}\bar{n} = \frac{1}{\sqrt{4^2+8^2+2^2}}(-4\hat{\imath} - 8\hat{\jmath} - 2\hat{k}) = \frac{1}{\sqrt{21}}(-2\hat{\imath} - 4\hat{\jmath} - \hat{k})$$ or negative

3. (6pts) The following equation determines a sphere. Find the center and radius of the sphere. **Hint:** Complete the squares.

   $$x^2 + y^2 + z^2 + 2x - 6y = -6$$

   6

   Center= (-1 , 3 , 0)

   Radius= 2

   $(x^2+2x) + (y^2-6y) + z^2 = -6$

   $(x^2+2x+1) + (y^2-6y+9) + z^2 = -6+1+9$

   $(x+1)^2 + (y-3)^2 + (z-0)^2 = 4 = 2^2$

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4. (6pts) Find a vector equation for a line that passes through the point $(1, 2, 3)$ at $t = 0$ and the point $(-1, 4, -2)$ at $t = 1$. Note, your answer should be a curve

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

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$P = (1,2,3)$

$Q = (-1,4,-2)$

$\overrightarrow{PQ} = \langle -1,4,-2\rangle - \langle 1,2,3\rangle = \langle -2,2,-5\rangle$

$\bar{r}(t) = \langle 1,2,3\rangle + t\langle -2,2,-5\rangle = \langle 1-2t, 2+2t, 3-5t\rangle$

5. (16pts) Suppose a particle's position at time t is given by the curve

$$\mathbf{r}(t) = 5t\mathbf{i} + \sin(3t)\mathbf{j} - \cos(3t)\mathbf{k}.$$

For this problem, it may be helpful if you remember the trig identity $\sin^2 x + \cos^2 x = 1$.

(a) (3pts) Find the velocity $\mathbf{v}(t)$ of the particle at time t.

3 $\quad \bar{v}(t) = \bar{r}'(t) = 5\hat{\imath} + 3\cos(3t)\hat{\jmath} + 3\sin(3t)\hat{k}$

(b) (3pts) Find the acceleration $\mathbf{a}(t)$ of the particle at time t.

3 $\quad \bar{a}(t) = \bar{v}'(t) = 0\hat{\imath} - 9\sin(3t)\hat{\jmath} + 9\cos(3t)\hat{k}$

(c) (3pts) Find the arc length of the curve between times $t = 0$ and $t = 3$.

$\|\bar{v}(t)\| = \sqrt{5^2 + 9\cos^2(3t) + 9\sin^2(3t)} = \sqrt{34}$

3 $\quad L = \int_0^3 \|\bar{v}(t)\|\, dt = \int_0^3 \sqrt{34}\, dt = 3\sqrt{34}$

(d) (3pts) Find the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{||\mathbf{v}(t)||}$.

3 $\quad \bar{T}(t) = \dfrac{\bar{v}(t)}{\|\bar{v}(t)\|} = \dfrac{1}{\sqrt{34}}\left(5\hat{\imath} + 3\cos(3t)\hat{\jmath} + 3\sin(3t)\hat{k}\right)$

(e) (4pts) Find the curvature $\kappa(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{v}(t)||}$ of the particle's path at time t.

$\bar{T}'(t) = \dfrac{1}{\sqrt{34}}\left(0\hat{\imath} - 9\sin(3t)\hat{\jmath} + 9\cos(3t)\hat{k}\right)$

$\|T'(t)\| = \dfrac{1}{\sqrt{34}}\sqrt{81\sin^2(3t) + 81\cos^2(3t)} = 9/\sqrt{34}$

4

$\kappa(t) = \dfrac{9/\sqrt{34}}{\sqrt{34}} = 9/34$

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6. (21pts, 1.5pts each) Match the equation with the type of surface it determines by writing the appropriate capital letter (**A-I**) in the provided blank. The last six equations are written in cylindrical or spherical coordinates. Some letters may be used more than once.

1.5 pts each.

C	$x^2 + y^2 - z^2 = 1$	**A** Elliptic Paraboloid
A	$x^2 + z^2 - y = 1$	**B** Ellipsoid
B	$x^2 + y^2 + \frac{z^2}{3} = 1$	**C** Hyperboloid of one sheet
D	$x^2 - 2y^2 - z^2 = 1$	**D** Hyperboloid of two sheets
E	$z = x^2 - y^2$	**E** Hyperbolic Paraboloid
G	$x^2 + y^2 + z^2 = 1$	**F** Plane
F	$x = 2y - z$	**G** Sphere
H	$x^2 + y^2 = 1$	**H** Cylinder
I	$r^2 = z^2$	**I** Cone
H	$r = 1$	
B	$2r^2 + z^2 = 1$	
I	$\phi = \frac{\pi}{5}$	
G	$\rho = 2$	
F	$\rho \cos \phi = 1$	

7. (9pts) Convert between Cartesian, cylindrical, and spherical coordinates as indicated. Please simplify as much as possible.

1pt each

(a) Find the spherical coordinates of the point with Cartesian coordinates $(-1, 1, \sqrt{2})$

$\rho =$ 2 $\qquad \theta =$ $3\pi/4$ $\qquad \phi =$ $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

not $\tan^{-1}(-1) = -\pi/4$

(b) Find the cylindrical coordinates of the point with Cartesian coordinates $(1, \sqrt{3}, -2)$

$r =$ 2 $\qquad \theta =$ $\tan^{-1}(\sqrt{3}) = \pi/3$ $\qquad z =$ -2

(c) Find the Cartesian coordinates of the point with spherical coordinates $(4, \frac{\pi}{3}, \frac{2\pi}{3})$

$x =$ $4\sin(2\pi/3)\cos\left(\frac{\pi}{3}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \sqrt{3}$

$y =$ $4\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 3$

$z =$ $4\cos(2\pi/3) = 4\left(-\frac{1}{2}\right) = -2$

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8. (12pts) Evaluate the following limits. If they do not exist, write 'DNE' and explain why.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos(x^2)}{\sqrt{1-x^2-y^2}}$ $= \dfrac{\cos(0)}{\sqrt{1-0^2-0^2}} = \dfrac{1}{1} = 1$

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(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+2y}{x-y}$

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If we approach $(0,0)$ along x-axis $(y=0)$, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+2y}{x-y} = \lim_{x\to 0} \frac{x}{x} = 1.$

If we approach $(0,0)$ along y-axis $(x=0)$, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+2y}{x-y} = \lim_{y\to 0} \frac{2y}{-y} = -2.$ So limit DNE

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-y}{x^2+y^2}$ **Hint:** Use polar coordinates.

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$\displaystyle= \lim_{r\to 0} \frac{-r\sin\theta}{r^2} = \lim_{r\to 0} \left(-\frac{1}{r}\right)\sin\theta$

This can be either $+\infty$ or $-\infty$ depending on sign of $\sin(\theta)$

So limit DNE

9. (13pts) Consider the function

$$f(x,y) = x^2\sin(xy) - x.$$

(a) (7pts) Find the equation of the tangent plane to the graph of $z = f(x,y)$ at the point $(1,0,-1)$.

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$f(1,0) = -1.$

$f_x(x,y) = 2x\sin(xy) + x^2y\cos(xy) - 1 \Rightarrow f_x(1,0) = -1.$

$f_y(x,y) = x^3\cos(xy) \Rightarrow f_y(1,0) = 1.$

$\boxed{z = -1 - 1(x-1) + 1(y-0)}$

$z = -x + y$

(b) (6pts) Find the following second derivatives:

2 i. $f_{xx}(x,y) =$ $2\sin(xy) + 4xy\cos(xy) - x^2y^2\sin(xy)$

2 ii. $f_{yy}(x,y) =$ $-x^4\sin(xy)$

2 iii. $f_{xy}(x,y) =$ $3x^2\cos(xy) - x^3y\sin(xy)$

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