2210-90 Exam 1
Fall 2012

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Please try to do all all work in the space provided. Page 5 is blank in case you need extra paper. Please circle your final answer.

1. (21pts) Consider the vectors $\mathbf{u} = \langle 1, 2, -3\rangle$ and $\mathbf{v} = \langle 1, 0, 2\rangle$. Find

   (a) (2pts) $\mathbf{u} - 5\mathbf{v}$

   $\langle 1,2,-3\rangle - \langle 5,0,10\rangle = \langle -4, 2, -13\rangle$

   (b) (2pts) $||\mathbf{u}||$

   $||u|| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

   (c) (2pts) The unit vector which points in the same direction as $\mathbf{u}$

   $\frac{1}{\sqrt{14}}\langle 1,2,-3\rangle = \langle 1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14}\rangle$

   (d) (2pts) $\mathbf{u}\cdot\mathbf{v}$

   $u\cdot v = (1)(1) + (2)(0) + (-3)(2) = 1 + 0 - 6 = -5$

   (e) (1pt) Are $\mathbf{u}$ and $\mathbf{v}$ orthogonal? Circle one: *YES* *NO*

   (f) (3pts) Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

   $-5 = \sqrt{14}\cdot\sqrt{5}\cos\theta \Rightarrow \theta = \cos^{-1}\left(-5/\sqrt{70}\right)$

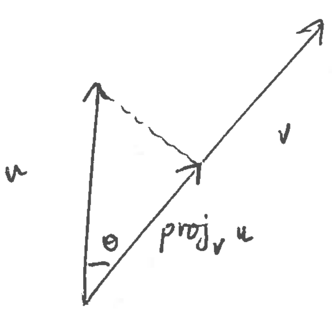
   (g) (3pts) $\mathbf{u}\times\mathbf{v}$

   $$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 1 & 0 & 2 \end{vmatrix} = \hat{i}(4) - \hat{j}(2+3) - 2k$$
   $$= \langle 4, -5, -2\rangle$$

   (h) (2pts) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{u})$

   $\langle 1,2,-3\rangle \cdot \langle -4,5,2\rangle = -4 + 10 - 6 = 0$

   (i) (4pts) Find the vector projection of $\mathbf{u}$ onto $\mathbf{v}$



$$\text{proj}_v u = \frac{||u||\cos\theta}{||v||}\cdot v \quad (2)$$
$$= \frac{\sqrt{14}\left(-5/\sqrt{70}\right)}{\sqrt{5}}\langle 1,0,2\rangle$$
$$= \langle -1, 0, -2\rangle \quad (2)$$

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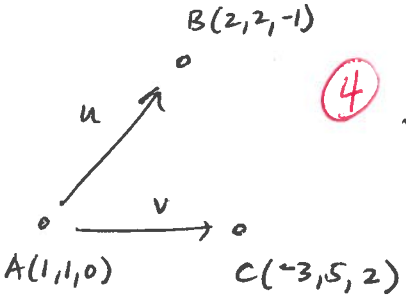
2. (10pts) Find an equation for the plane which contains the points $(1,1,0)$, $(2,2,-1)$, and $(-3,5,2)$.



(4) $u = \langle 2,2,-1\rangle - \langle 1,1,0\rangle = \langle 1,1,-1\rangle.$

$v = \langle -3,5,2\rangle - \langle 1,1,0\rangle = \langle -4,4,2\rangle.$

wrong vectors (-3)

other vectors possible

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -4 & 4 & 2 \end{vmatrix} = \hat{i}(6) - \hat{j}(-2) + 8\hat{k}$$

(3)

$$6x + 2y + 8z = 8$$

(3)

3. (8pts) Find the equation for the sphere which is centered at $(-3,4,1)$ and contains the point (1,0,-1).

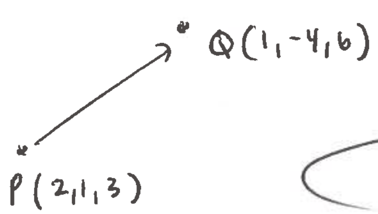
$$(x+3)^2 + (y-4)^2 + (z-1)^2 = r^2$$ (4)

plug in $(x,y,z) = (1,0,-1)$

$$(1+3)^2 + (0-4)^2 + (-1-1)^2 = 4^2 + 4^2 + 2^2 = 36$$ (4)

$$(x+3)^2 + (y-4)^2 + (z-1)^2 = 36$$

4. (8pts) Find a parametric equation for a line $\mathbf{r}(t)$ that passes through the point $(2,1,3)$ at $t = 0$ and passes through the point $(1,-4,6)$ at $t = 1$.



$$|\overrightarrow{PQ}| = \langle 1,-4,6\rangle - \langle 2,1,3\rangle$$
$$= \langle -1,-5,3\rangle$$ (4)

$$r(t) = \langle 2,1,3\rangle + t\,\langle -1,-5,3\rangle$$ (4)

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5. (8pts) Find the arclength of the curve

$$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^3\mathbf{j} + \frac{\sqrt{2}}{2}t^2\mathbf{k}.$$

for $0 \leq t \leq 2$.

$r'(t) = \hat{i} + t^2\hat{j} + \sqrt{2}t\hat{k}$. 2

$\|r'(t)\| = \sqrt{1 + t^4 + 2t^2} = \sqrt{(t^2+1)^2} = t^2+1$. 2

$L = \int_0^2 \|r'(t)\| dt = \int_0^2 (t^2+1) dt = \left(\frac{1}{3}t^3 + t\right|_0^2 = \frac{8}{3} + 2 = \frac{14}{3}$ 2 2

6. (12pts) Suppose a particle's position at time t is given by the curve

$$\mathbf{r}(t) = t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}.$$

(a) (2pts) Find the velocity $\mathbf{v}(t)$ of the particle at time t.

$v(t) = \hat{i} + \cos t\,\hat{j} - \sin t\,\hat{k}$ 2

(b) (2pts) Find the acceleration $\mathbf{a}(t)$ of the particle at time t.

$a(t) = -\sin t\,\hat{j} - \cos t\,\hat{k}$ 2

(c) (6pts) Find the curvature $\kappa(t)$ of the particle's path at time t.

$\|r'(t)\| = \|v(t)\| = \sqrt{1^2 + \cos^2 t + \sin^2 t} = \sqrt{2}$. 2

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = -\hat{i} + \cos t\,\hat{j} - \sin t\,\hat{k}$$ 2

$\|r' \times r''\| = \sqrt{2}$.

other methods work too

$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{1}{2}$. 2

(d) (2pts) For all time t, this curve is contained on what type of surface? Circle the correct letter

A a sphere of radius 1 centered at the origin

B a cylinder of radius 1 parallel to the z-axis

C a paraboloid opening in the positive z direction

Not correct

+2 for everyone

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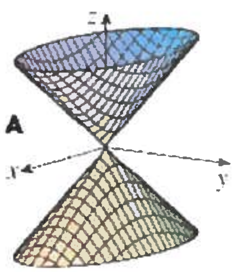
7. (18 pts) Match the equation with the type of surface it describes by writing the appropriate capital letter (**A-F**) in the provided blank (**Note:** The last three equations are written using cylindrical coordinates).

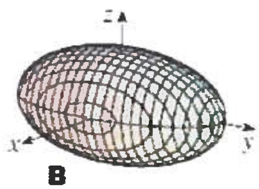
2 pts each

(a) D $z = y^2 - x^2$
(b) A $z^2 = x^2 + y^2$
(c) B $4x^2 + y^2 + 4z^2 = 1$
(d) F $-x^2 - 5y^2 + z^2 = 1$
(e) C $z = 5x^2 + y^2$
(f) E $3x^2 + y^2 - 5z^2 = 3$
(g) C $z = r^2$
(h) A $|z| = r$
(i) E $r^2 - z^2 = 1$

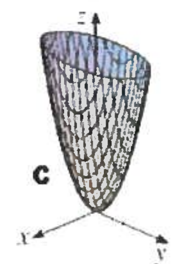
Cone



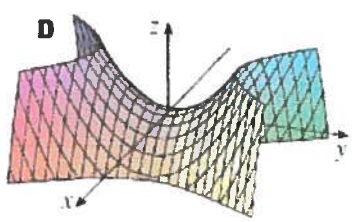
Ellipsoid



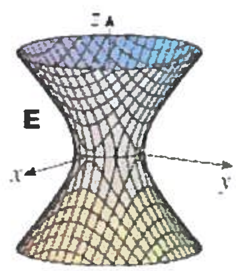
Elliptic Paraboloid



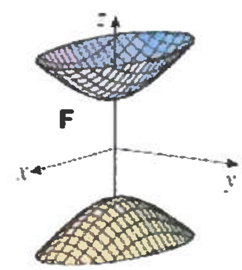
Hyperbolic Paraboloid



Hyperboloid of One Sheet



Hyperboloid of Two Sheets



8. (15pts) Convert between Cartesian, cylindrical, and spherical coordinates as indicated

1 pt each

(a) Find the cylindrical coordinates of the point with Cartesian coordinates $(2, -2, 3)$

$r =$ $\sqrt{8}$ $\theta =$ $-\pi/4$ $z =$ 3

(b) Find the spherical coordinates of the point with Cartesian coordinates $(0, 5, 0)$

$\rho =$ 5 $\theta =$ $\pi/2$ $\phi =$ $\pi/2$

(c) Find the Cartesian coordinates of the point with spherical coordinates $(3, \frac{\pi}{6}, \frac{3\pi}{4})$

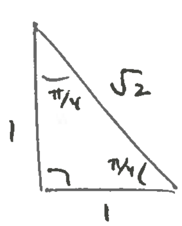
$x =$ $3\sin(\frac{3\pi}{4})\cos(\frac{\pi}{6}) = \frac{3\sqrt{6}}{4}$ $y =$ $3\sin(\frac{3\pi}{4})\sin(\frac{\pi}{6}) = \frac{3\sqrt{2}}{4}$ $z =$ $3\cos(\frac{3\pi}{4}) = -\frac{3\sqrt{2}}{2}$

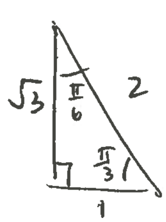
(d) Find the Cartesian coordinates of the point with cylindrical coordinates $(2, -\frac{\pi}{3}, -5)$

$x =$ $2\cos(-\frac{\pi}{3}) = 1$ $y =$ $2\sin(-\pi/3) = -\sqrt{3}$ $z =$ -5

(e) Find the spherical coordinates of the point with cylindrical coordinates $(1, -\frac{\pi}{4}, 0)$

$\rho =$ 1 $\theta =$ $-\pi/4$ $\phi =$ $\pi/2$





$\sin(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}$

$\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ $\sin(\frac{\pi}{6}) = \frac{1}{2}$

cartesian $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)$

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