Calculus III 2210-90 Final Exam

Summer 2014

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Correct answers unaccompanied by work may not receive full credit. Please try to do all all work in the space provided and circle your final answer.

1. (13pts) Consider the vectors $\mathbf{u} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

   (a) (3pts) Find $\mathbf{u} + \mathbf{v}$

   3

   $$= 6\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

   (b) (3pts) Find $\mathbf{u} \cdot \mathbf{v}$

   3

   $$= (1)(5) + (-1)(2) + (3)(-1) = 5 - 2 - 3 = 0.$$

   (c) (3pts) Find the angle between $\mathbf{u}$ and $\mathbf{v}$. Give your answer in radians.

   3

   Since $u \cdot v = 0$, $\theta = \frac{\pi}{2}$ (or $\cos^{-1}(0)$).

   (d) (4pts) Find $\mathbf{u} \times \mathbf{v}$

   4

   $$\bar{u} \times \bar{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 3 \\ 5 & 2 & -1 \end{vmatrix} = \hat{\imath}(1-6) + \hat{\jmath}(15+1) + \hat{k}(2+5)$$

   $$= -5\hat{\imath} + 16\hat{\jmath} + 7\hat{k}$$

2. (10pts) Consider the function $f(x, y) = \frac{x}{1+y^2}$

   (a) (6pts) Find the equation of the tangent plane to $z = f(x, y)$ at the point $(2, 1, 1)$.

   6

   $$f_x(x,y) = \frac{1}{1+y^2} \Rightarrow f_x(2,1) = \frac{1}{2}.$$

   $$f_y(x,y) = \frac{-x(2y)}{(1+y^2)^2} \Rightarrow f_y(2,1) = \frac{-4}{4} = -1.$$

   $$z = 1 + \frac{1}{2}(x-2) - 1(y-1)$$

   $\frac{1}{2}x - 1 - y + 1 + 1 = z$

   $\frac{1}{2}x - y + 1 = z$

   $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

   (b) (4pts) Find a unit vector which is perpendicular to the tangent plane to $z = f(x, y)$ at the point $(5, 2, 1)$

   4

   A normal vector is of the form

   $$\langle f_x, f_y, -1 \rangle.$$

   So at $(5,2,-1)$

   $$\langle f_x(5,2), f_y(5,2), -1 \rangle = \langle \frac{1}{5}, \frac{-4}{5}, -1 \rangle$$ 3

   not a unit vector, so divide by length

   $$= \frac{1}{\sqrt{\frac{1}{25} + \frac{16}{25} + 1}} \langle \frac{1}{5}, \frac{-4}{5}, -1 \rangle$$

23

3. (12pts) Match the equation with the graph of the surface it describes by writing the appropriate capital letter (**A-F**) in the provided blank. Each answer will be used exactly once.

12

(a) F $x^2 - y - z^2 = 0$

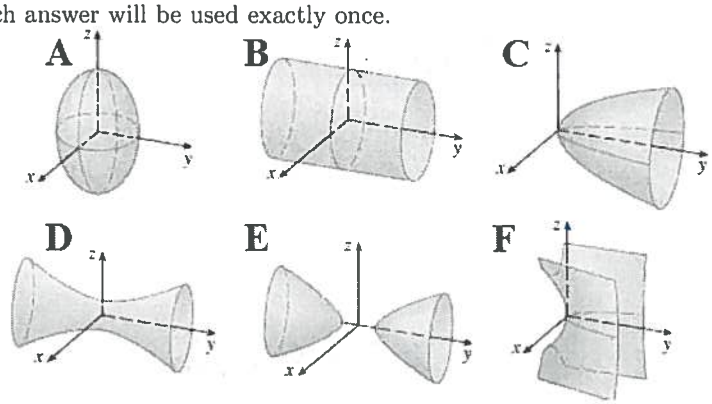
(b) E $x^2 - y^2 + z^2 = -1$

(c) D $x^2 - y^2 + z^2 = 1$

(d) C $x^2 - y + z^2 = 0$

(e) A $4x^2 + 4y^2 + z^2 = 4$

(f) B $x^2 + z^2 = 2$



4. (10pts) Use Lagrange multipliers to find the extreme values (both maximum and minimum) of the function $f(x, y) = y^2 - x$ on the circle $(x - 1)^2 + y^2 = 1$.

$f(x,y) = y^2 - x \Rightarrow \nabla f(x,y) = \langle -1, 2y \rangle.$

$0 = g(x,y) = (x-1)^2 + y^2 - 1 \Rightarrow \nabla g(x,y) = \langle 2x-2, 2y \rangle$

10

So LM equs: 6

① $-1 = 2\lambda(x-1)$

② $2y = 2\lambda y$

③ $(x-1)^2 + y^2 = 1$

Equ ② $\Rightarrow \lambda = 1$ or $y = 0$

if $y=0$, then ③ $\rightarrow (x-1)^2 = 1 \Rightarrow x = 0, 2.$

if $\lambda = 1$, then ① $\Rightarrow -1 = 2x - 2 \Rightarrow x = \frac{1}{2}$

then ③ $\rightarrow \frac{1}{4} + y^2 = 1 \Rightarrow y = \pm\frac{\sqrt{3}}{2}.$

So my points are $(0,0), (2,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{1}{2}, -\frac{\sqrt{3}}{2})$

$f(0,0) = 0$

$f(2,0) = -2$ min

$f(\frac{1}{2}, \frac{\sqrt{3}}{2}) = f(\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ max

5. (26pts) Evaluate the following double or triple integrals.

(a) (5pts) $\iint_R (x^2 + y)\, dA$, where R is the rectangle $0 \le x \le 1$, $1 \le y \le 3$.

$$= \int_1^3 \int_0^1 (x^2+y)\,dx\,dy = \int_1^3 \left(\frac{x^3}{3} + yx \Big|_0^1\right) dy = \int_1^3 \left(\frac{1}{3} + y\right) dy$$

5

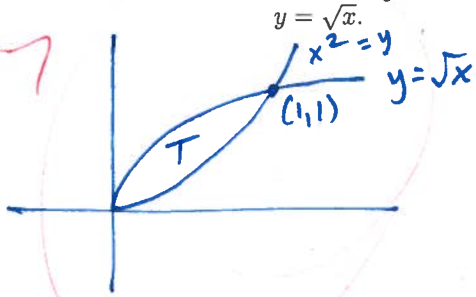
$$= \left(\frac{y}{3} + \frac{y^2}{2}\right|_1^3 = \left(\frac{3}{3} + \frac{9}{2}\right) - \left(\frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{33}{6} - \frac{5}{6} = \frac{28}{6} = \frac{14}{3}$$

27

(b) (7pts) $\iint_T xy\, dA$, where T is the region in the first quadrant between the curves $y = x^2$ and $y = \sqrt{x}$.

7



$$\iint_T xy\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} xy\, dy\, dx$$

5

$$= \int_0^1 \left(x \frac{y^2}{2} \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 \left(\frac{x^2}{2} - \frac{x^5}{2} \right) dx \right.$$

$$= \left(\frac{x^3}{6} - \frac{x^6}{12} \Big|_0^1 = \frac{1}{6} - \frac{1}{12} = \frac{1}{12} \right.$$

(c) (7pts) $\iint_D \frac{1}{1 + x^2 + y^2}\, dA$, where D is the unit disk $x^2 + y^2 \leq 1$.

Use polar coordinates.

7

$$= \int_0^{2\pi} \int_0^1 \frac{1}{1+r^2} r\, dr\, d\theta = 2\pi \int_0^1 \frac{r}{1+r^2}\, dr = \pi \int_1^2 \frac{1}{u}\, du$$

$u = 1 + r^2$
$du = 2r\, dr$

$$= \pi \left(\ln u \right|_1^2$$

$$= \pi (\ln 2)$$

(d) (7pts) $\iiint_S z^2\, dV$, where S is the sphere $x^2 + y^2 + z^2 \leq 4$.

Use spherical coords:

7

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^2 \rho^2 \cos^2\phi \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi \left(\int_0^{\pi} \cos^2\phi \sin\phi\, d\phi \right) \left(\int_0^2 \rho^4\, d\rho \right)$$

$$= 2\pi \left(-\frac{1}{3} \cos^3\phi \Big|_0^{\pi} \right) \left(\frac{\rho^5}{5} \Big|_0^2 \right) = 2\pi \left(\frac{1}{3} + \frac{1}{3} \right) \left(\frac{32}{5} \right) = 2\pi \left(\frac{2}{3} \right) \left(\frac{32}{5} \right)$$

$$= \frac{128\pi}{15}$$

6. (10pts) Circle T if the statement is true and F if the statement is false in general. Let $\mathbf{F}(x, y)$ denote a vector field and C an oriented curve.

T	F	The value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same regardless of how you parameterize the curve C.
T	F	There is always a function $f(x, y)$ such that $\mathbf{F}(x, y) = \nabla f(x, y)$.
T	F	If $\operatorname{div} \mathbf{F} = 0$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.
T	F	If $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F}$ is conservative.
T	F	If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, then C must be a circle.

10

31

7. (18pts) Consider the vector field $\mathbf{F}(x,y) = (e^y + 8x)\mathbf{i} + xe^y\mathbf{j}$.

(a) (4pts) Compute div $\mathbf{F}$

4

$$\operatorname{div} F = \frac{\partial}{\partial x}(e^y + 8x) + \frac{\partial}{\partial y}(xe^y) = 8 + xe^y$$

(b) (4pts) Compute curl $\mathbf{F}$

4

$$\operatorname{curl} F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^y + 8x & xe^y & 0 \end{vmatrix} = (e^y - e^y)\hat{k} = \bar{0}.$$

2

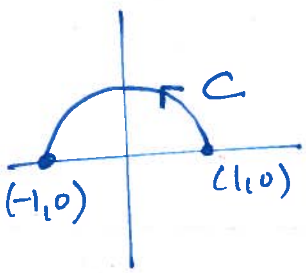
(c) (2pts) Is $\mathbf{F}$ a conservative vector field? Circle one: YES NO

(d) (8pts) If C is the curve parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le \pi$, use the Fundamental Theorem of Line Integrals to compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

8

Find a potential function: if $f(x,y) = xe^y + 4x^2$, $+ c$ then $\nabla f(x,y) = F(x,y)$.



By FTLI:

$$\int_C F \cdot dr = f(-1,0) - f(1,0)$$
$$= (-1+4) - (1+4) = -2$$

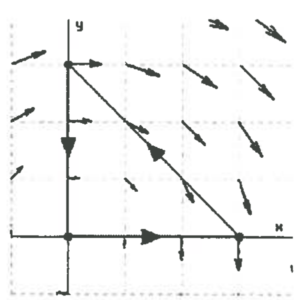
8. (6pts) Let C be the curve and $\mathbf{F}$ the vector field given by the picture below. Indicate whether each statement is true (T) or false (F) by writing in the blank provided.

6

T $\int_C \mathbf{F} \cdot d\mathbf{r} < 0$

T $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$

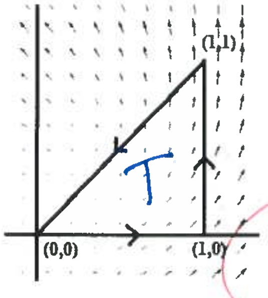
F There is a function f such that $\mathbf{F} = \nabla f$.



24.

$$\int_C F \cdot dr = \iint_R \text{curl } F \cdot \hat{k}\, dA$$

9. (16pts) Let $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x^2 + y^2)\mathbf{j}$ and let C denote the boundary of the triangle with vertices at $(0, 0)$, $(1, 0)$, and $(1, 1)$, traversed counterclockwise. See picture below.



(a) (8pts) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Green's Theorem.

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x-y & x^2+y^2 & 0 \end{vmatrix} = \hat{k}(2x+1)$$

$$\int_C F \cdot dr = \iint_T (2x+1)\, dA = \int_0^1 \int_0^x (2x+1)\, dy\, dx = \int_0^1 (2xy + y \Big|_0^x dx)$$

$$= \int_0^1 (2x^2 + x)\, dx = \left(\frac{2}{3}x^3 + \frac{x^2}{2}\right|_0^1 = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}$$

8 5 2 1

(b) (8pts) Compute the line integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$ using Green's Theorem (this version is also called the Plane Divergence Theorem).

$$\text{div } F = \partial/\partial x\,(x-y) + \frac{\partial}{\partial y}(x^2+y^2) = 1 + 2y.$$

$$\int_C F \cdot n\, ds = \iint_T (1+2y)\, dA = \int_0^1 \int_0^x (1+2y)\, dy\, dx = \int_0^1 (y + y^2 \Big|_0^x dx$$

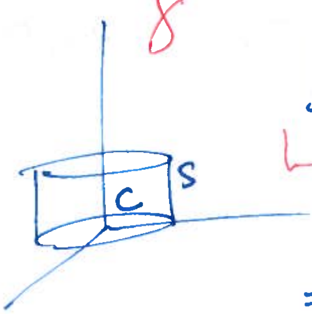
$$= \int_0^1 (x + x^2)\, dx = \left(\frac{x^2}{2} + \frac{x^3}{3}\right|_0^1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

8 5 2 1

10. (8pts) Use the Divergence Theorem to compute the flux of the vector field $\mathbf{F}(x, y, z) = xy^2z\mathbf{i} - x^3\mathbf{j} + \frac{1}{2}x^2z^2\mathbf{k}$ through the boundary of the cylinder determined by $x^2 + y^2 \leq 1, 0 \leq z \leq 1$.

$$\text{div } F = y^2 z + x^2 z = z(x^2+y^2)$$

8



$$\iint_S F \cdot n\, dS = \iiint_C \text{div } F\, dV = \iiint_C z(x^2+y^2)\, dV$$

Use cylindrical coords:

$$= \int_0^{2\pi} \int_0^1 \int_0^1 z r^2 r\, dr\, dz\, d\theta = 2\pi \left(\int_0^1 z\, dz\right)\left(\int_0^1 r^3\, dr\right)$$

$$= 2\pi \left(\frac{z^2}{2}\right|_0^1 \left(\frac{r^4}{4}\right|_0^1$$

$$= 2\pi\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{\pi}{4}$$

24

11. (11pts) Consider the surface G in 3-space determined by the parametric equations

$$\mathbf{r}(u,v) = (u+v)\mathbf{i} + (u-v)\mathbf{j} + (2u+7v)\mathbf{k},$$

where $0 \le u \le 1, 0 \le v \le 1$.

(a) (6pts) Compute $||\mathbf{r}_u \times \mathbf{r}_v||$.

$$\bar{r}_u = \hat{\imath} + \hat{\jmath} + 2\hat{k} \qquad \bar{r}_v = \hat{\imath} - \hat{\jmath} + 7\hat{k}.$$

6

$$\bar{r}_u \times \bar{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 2 \\ 1 & -1 & 7 \end{vmatrix} = \hat{\imath}(9) + \hat{\jmath}(-5) + \hat{k}(-2).$$

$$\|\bar{r}_u \times \bar{r}_v\| = \sqrt{9^2+5^2+2^2} = \sqrt{110}$$

(b) (5pts) Compute the surface integral $\iiint_G (x+y)\, dS$.

Recall: For surfaces defined parametrically, $dS = ||\mathbf{r}_u \times \mathbf{r}_v||\, du\, dv$

$$\iiint_G (x+y)\, dS = \int_0^1 \int_0^1 \big((u+v)+(u-v)\big)\sqrt{110}\, du\, dV$$

5

$$= \sqrt{110}\int_0^1\int_0^1 2u\, du\, dV = \sqrt{110}.$$

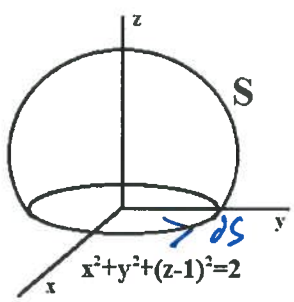
12. (10pts) Let S denote the surface determined by $x^2+y^2+(z-1)^2 = 2$ where $z > 0$ (S is the part of the sphere of radius $\sqrt{2}$ centered at $(0,0,1)$ with positive z component...see picture below). Use Stokes Theorem to evaluate

$$\iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the outward pointing unit normal and $\mathbf{F}$ denotes the vector field

$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}.$$

10



The bandary of S can be parametrized by

$r(t) = \langle \cos t, \sin t, 0 \rangle. \Rightarrow r'(t) = \langle -\sin t, \cos t, 0 \rangle$

$0 \le t \le 2\pi$

By Stokes'

$$\iiint_S \text{curl}\,F \cdot n\, dS = \int_{\partial S} F \cdot dr$$

$$= \int_0^{2\pi} (-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 0\hat{k}) \cdot (-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi$$

21