2210-90 Final Exam

Spring 2014

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Correct answers unaccompanied by work may not receive full credit. Please try to do all all work in the space provided and circle your final answer.

1. (15pts) Consider the vectors $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, and $\mathbf{w} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

   (a) (3pts) Find $\mathbf{u} + \mathbf{v} - \mathbf{w}$

   3

   $= (2\hat{\imath} - \hat{\jmath} + \hat{k}) + (-\hat{\imath} + \hat{\jmath} + 2\hat{k}) - (4\hat{\imath} + 2\hat{\jmath} + \hat{k}) = -3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

   (b) (3pts) Find $||\mathbf{w}||$

   3

   $\|\vec{w}\| = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$

   (c) (3pts) Find $\mathbf{u} \cdot \mathbf{v}$

   3

   $(2\hat{\imath} - \hat{\jmath} + \hat{k}) \cdot (-\hat{\imath} + \hat{\jmath} + 2\hat{k}) = (2)(-1) + (-1)(1) + (1)(2) = -1$

   (d) (2pts) Which two vectors are orthogonal? Circle one:

   2

   A. **u, v** B. **u, w** C. **v, w**

   (e) (4pts) Find $\mathbf{u} \times \mathbf{w}$

   4

   $$\bar{u} \times \bar{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = \hat{\imath}(-1-2) - \hat{\jmath}(2-4) + \hat{k}(4+4)$$

   $$= -3\hat{\imath} + 2\hat{\jmath} + 8\hat{k}$$

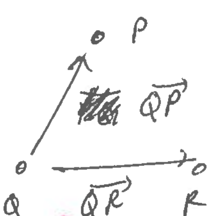
2. (8pts) Find an equation for the plane which contains the points $(0, 2, 3)$, $(1, -1, 3)$ and $(1, -2, 1)$.

   P Q R

   8

   $\overrightarrow{QP} = \langle 0,2,3 \rangle - \langle 1,-1,3 \rangle = \langle -1,3,0 \rangle$. 1

   $\overrightarrow{QR} = \langle 1,-2,1 \rangle - \langle 1,-1,3 \rangle = \langle 0,-1,-2 \rangle$ 1

   $$\bar{n} = \overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 3 & 0 \\ 0 & -1 & -2 \end{vmatrix} = \hat{\imath}(-6) - \hat{\jmath}(2) + \hat{k}(1)$$

   $$= -6\hat{\imath} - 2\hat{\jmath} + \hat{k}. \quad 3$$

   $-6x - 2y + z = d$

   Plane contains $(0,2,3)$

   $-6(0) - 2(2) + 3 = -1$ 3

   $-6x - 2y + z = -1.$

3. (15pts) Consider the function $f(x,y) = \sqrt{x^2+y^2}$. $= (x^2+y^2)^{1/2}$

(a) (5pts) Find $\nabla f(x,y)$, the gradient of $f(x,y)$.

5

$$f_x = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}}$$

$$\nabla f(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

(b) (5pts) Find the equation of the tangent plane to the graph of $z = f(x,y)$ at the point $(3,4,5)$.

$$f(3,4) = \sqrt{3^2+4^2} = 5.$$

5

$$\nabla f(3,4) = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$$z = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$$

(c) (5pts) Use part (b) above to estimate the value of $\sqrt{(2.9)^2+(4.2)^2}$.

Since $2.9 \approx 3$ and $4.2 \approx 4$

5

$$\sqrt{(2.9)^2+(4.2)^2} = f(2.9, 4.2) \approx 5 + \frac{3}{5}(2.9-3) + \frac{4}{5}(4.2-4)$$

$$= 5 + \left(\frac{3}{5}\right)(-.1) + \left(\frac{4}{5}\right)(.2) = 5 + \frac{1}{10} = 5.1$$

4. (12pts) Use Lagrange multipliers to find the extreme values (both maximum and minimum) of the function $f(x,y) = x^2+y^2$ on the circle $(x-1)^2+y^2 = 1$.

$$f(x,y) = x^2+y^2$$

$$g(x,y) = (x-1)^2+y^2-1 = 0.$$

12

$$\nabla f = \lambda \nabla g, \quad g = 0 \implies \begin{cases} (1)\ 2x = 2\lambda(x-1) \\ (2)\ 2y = 2\lambda y \\ (3)\ (x-1)^2+y^2-1 = 0 \end{cases}$$

6

(2) $\Rightarrow$ $\lambda = 1$ or $y = 0$.

If $\lambda = 1$, then (1) $\Rightarrow$ $2x = 2x - 2$, which has no solution.

If $y = 0$, then (3) $\Rightarrow$ $(x-1)^2 = 1$ $\Rightarrow$ $x = 2, 0$.

~~This~~ gives the points $(0,0)$ and $(2,0)$.

2 2

$f(0,0) = 0$ min value

$f(2,0) = 4$ max value

2

27

5. (26pts) Evaluate the following double or triple integrals:

(a) (5pts) $\iint_R xy\, dA$, where R is the rectangle $0 \le x \le 2$, $2 \le y \le 4$.

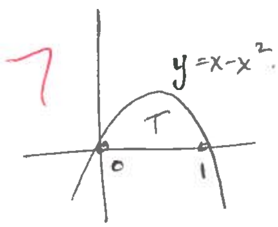
5

$$= \int_2^4 \int_0^2 xy\, dx\, dy = \int_2^4 \left(\frac{x^2}{2} y \Big|_0^2 \right) dy = \int_2^4 2y\, dy = \left(y^2 \Big|_2^4 \right.$$

$$= 16 - 4 = 12$$

(b) (7pts) $\iint_T (x + 2y)\, dA$, where T is the region bounded by the curve $y = x - x^2$ and the x-axis.

7



$$= \int_0^1 \int_0^{x-x^2} (x+2y)\, dy\, dx = \int_0^1 \left(xy + y^2 \Big|_0^{x-x^2} \right. dx = \int_0^1 \left(x(x-x^2) + (x-x^2)^2 \right) dx$$

$$= \int_0^1 (2x^2 - 3x^3 + x^4)\, dx = \left(\frac{2}{3}x^3 - \frac{3}{4}x^4 + \frac{x^5}{5} \Big|_0^1 \right. = \frac{2}{3} - \frac{3}{4} + \frac{1}{5} = \frac{7}{60}$$

(c) (7pts) $\iint_S x\, dA$, where S is the piece of the disk $x^2 + y^2 \le 1$ in the first quadrant.

Use polar coords (or not)

7

$$= \int_0^{\pi/2} \int_0^1 (r\cos\theta) r\, dr\, d\theta = \int_0^{\pi/2} \cos\theta \left(\frac{r^3}{3} \Big|_0^1 \right. d\theta = \frac{1}{3} \int_0^{\pi/2} \cos\theta\, d\theta =$$

$$= \frac{1}{3} \left(\sin\theta \Big|_0^{\pi/2} \right. = \frac{1}{3}$$

(d) (7pts) $\iiint_C z\sqrt{x^2 + y^2}\, dV$, where C is the cylinder $x^2 + y^2 \le 1$, $0 \le z \le 2$.

Use cylindrical coords:

7

$$= \int_0^2 \int_0^{2\pi} \int_0^1 (zr) r\, dr\, d\theta\, dz = \int_0^2 \int_0^{2\pi} z \left(\frac{r^3}{3} \Big|_0^1 \right. d\theta\, dz$$

$$= \frac{1}{3} \int_0^2 \int_0^{2\pi} z\, d\theta\, dz$$

$$= \frac{2\pi}{3} \int_0^2 z\, dz$$

$$= \frac{2\pi}{3} \left(\frac{z^2}{2} \Big|_0^2 \right. = \frac{4\pi}{3}$$

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6. (10pts) Match the following vector fields with the level curves to which they are perpendicular by writing the letter in the blank provided.

10

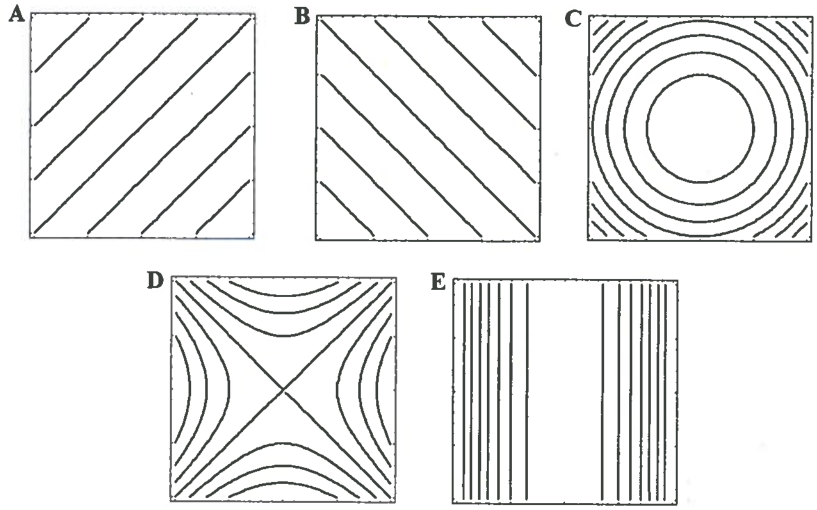
C $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

A $\mathbf{F}(x, y) = -\mathbf{i} + \mathbf{j}$

B $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

D $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$

E $\mathbf{F}(x, y) = x\mathbf{i}$



7. (18pts) Consider the vector field $\mathbf{F}(x, y) = (x^2 + y)\mathbf{i} + y\mathbf{j}$.

(a) (4pts) Compute div $\mathbf{F}$

4

$$\text{div } F = \frac{\partial}{\partial x}(x^2+y) + \frac{\partial}{\partial y}(y) = 2x + 1.$$

(b) (4pts) Compute curl $\mathbf{F}$

4

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y & y & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x^2+y)\right) = -\hat{k}.$$

2

(c) (2pts) Is $\mathbf{F}$ a conservative vector field? Circle one: YES NO

(d) (6pts) If C_1 is the curve parameterized by $\mathbf{r}(t) = \langle t, \sin(\pi t)\rangle$ for $0 \leq t \leq 1$, compute

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

6

$$r(t) = \langle t, \sin(\pi t)\rangle$$
$$r'(t) = \langle 1, \pi\cos(\pi t)\rangle.$$

$$\int_{C_1} F \cdot dr = \int_0^1 \left[(t^2 + \sin(\pi t))\hat{i} + \sin(\pi t)\hat{j}\right] \cdot \left[\hat{i} + \pi\cos(\pi t)\hat{j}\right] dt$$

$$= \int_0^1 \left(t^2 + \sin(\pi t) + \pi\sin(\pi t)\cos(\pi t)\right)dt = \left(\frac{t^3}{3} - \frac{1}{\pi}\cos(\pi t) + \frac{1}{2}\sin^2(\pi t)\right)\Big|_0^1$$

$$= \frac{1}{3} + \frac{1}{\pi} + \frac{1}{\pi} = \frac{1}{3} + \frac{2}{\pi}$$

(e) (2pts) If C_2 is another curve connecting $(0, 0)$ to $(1, 0)$, would you expect $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ to be equal to $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$? Circle one: YES NO

2

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8. (16pts) Let $\mathbf{F}(x, y) = (3x + y^3)\mathbf{i} + (y - x^3)\mathbf{j}$ and let C denote the unit circle traversed counterclockwise.

(a) (8pts) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Green's Theorem.

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x+y^3 & y-x^3 & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial}{\partial x}(y-x^3) - \frac{\partial}{\partial y}(3x+y^3)\right)$$

$$= (-3x^2 - 3y^2)\hat{k}.$$

8

$$\int_C F \cdot dr \underset{\substack{\uparrow \\ \text{Green's Thm}}}{=} \iint_S (-3x^2-3y^2)\, dA \underset{\substack{\uparrow \\ \text{polar coords}}}{=} \int_0^{2\pi}\int_0^1 -3r^3\, dr\, d\theta$$

$$= -6\pi\left(\frac{r^4}{4}\Big|_0^1\right) = -\frac{3\pi}{2}$$

(b) (8pts) Compute the line integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$ using Green's Theorem (this version is also called the Plane Divergence Theorem).

$$\text{div } F = \frac{\partial}{\partial x}(3x+y^3) + \frac{\partial}{\partial y}(y-x^3) = 3+1=4.$$

8

$$\int_C F \cdot n\, ds \underset{\substack{\uparrow \\ \text{Green's Thm}}}{=} \iint_S 4\, dA = 4A(S) = 4\pi$$

9. (8pts) Consider the vector field $\mathbf{F}(x, y, z) = (2xz + y)\mathbf{i} + (x^2 - 3y)\mathbf{j} + (3z - z^2 + y)\mathbf{k}$. Use the Divergence Theorem to compute

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where S is the surface of the ellipse $2x^2 + y^2 + z^2 = 1$, and $\mathbf{n}$ is the outward pointing unit normal vector.

$$\text{div } F = \frac{\partial}{\partial x}(2xz+y) + \frac{\partial}{\partial y}(x^2-3y) + \frac{\partial}{\partial z}(3z-z^2+y)$$

8

$$= 2z - 3 + (3 - 2z)$$

$$= 0.$$

$$\iint_S F \cdot n\, dS \underset{\substack{\uparrow \\ \text{Div. Thm}}}{=} \iiint_E \text{div } F\, dV = \iiint_E 0\, dV$$

$$= 0$$

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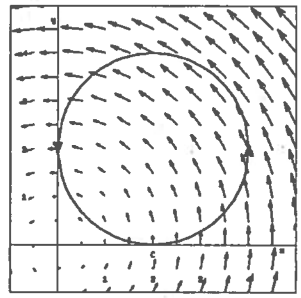
10. (6pts) Let C be the curve and vector field given by the picture below. Indicate whether each statement is true (T) or false (F) by writing in the blank provided.

F $\int_C \mathbf{F} \cdot d\mathbf{r} < 0$

T $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$

F **F** is conservative.

6



11. (16pts) Let S denote the surface determined by

$$z = x^2 - y^2,$$

where $x^2 + y^2 \leq 1$ (S is the graph of $f(x, y) = x^2 - y^2$ over the unit disk).

(a) (8pts) Find the surface area of S.

$$f(x,y) = x^2 - y^2.$$

$$f_x(x,y) = 2x \quad , \quad f_y(x,y) = -2y$$

8

$$A(S) = \iint_D \sqrt{1 + (2x)^2 + (-2y)^2}\, dA = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA$$

Use polar coords:

$$= \int_0^{2\pi} \int_0^1 \sqrt{1+4r^2}\, r\, dr\, d\theta = 2\pi \int_0^1 \sqrt{1+4r^2}\, r\, dr = \frac{\pi}{4} \int_1^5 u^{1/2}\, du$$

$u = 1 + 4r^2$

$du = 8r\, dr$

$$= \frac{\pi}{4} \left(\frac{2}{3} u^{3/2} \Big|_1^5 \right.$$

$$= \frac{\pi}{6} \left(5^{3/2} - 1 \right)$$

(b) (8pts) Use Stokes Theorem to evaluate

$$\iint_S (\text{curl}\, \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where **F** denotes the vector field

$$\mathbf{F} = (x^2 + y^2 - 1)\mathbf{i} + (z - x^2 + y^2)\mathbf{k}.$$

8

Hint: What does **F** look like on the boundary of S (the graph of $z = x^2 - y^2$ over the circle $x^2 + y^2 = 1$)?

On the bondary of S, $z = x^2 - y^2$ and $x^2 + y^2 = 1$,

So when (x, y, z) is on the bondary of S, $F(x,y,z) = 0\hat{i} + \vec{0}$
$= \vec{0}$

Stokes Thm says

$$\iint_S (\text{curl}\, F) \cdot n\, dS = \int_{\partial S} F \cdot dr = \int_{\partial S} 0\, ds = 0$$

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