2210-90 Final Exam

Spring 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Correct answers unaccompanied by work may not receive full credit. Please try to do all all work in the space provided and circle your final answer.

1. (16pts) Let $\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

   (a) (4pts) Find $2\mathbf{u} + \mathbf{v}$

   4

   $$2(\hat{\imath} + 3\hat{\jmath} - 2\hat{k}) + (\hat{\imath} + \hat{\jmath} - \hat{k}) = 3\hat{\imath} + 7\hat{\jmath} - 5\hat{k}$$

   (b) (4pts) Find $\|\mathbf{u}\|$

   4

   $$= \sqrt{1^2 + 3^2 + 2^2} = \sqrt{1 + 9 + 4} = \sqrt{14}$$

   (c) (4pts) Find $\mathbf{u} \cdot \mathbf{v}$

   4

   $$u \cdot v = (1)(1) + (3)(1) + (-2)(-1) = 6$$

   (d) (4pts) Find $\mathbf{v} \times \mathbf{u}$

   4

   $$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix} = \hat{\imath}(-2+3) + \hat{\jmath}(-1+2) + \hat{k}(3-1)$$

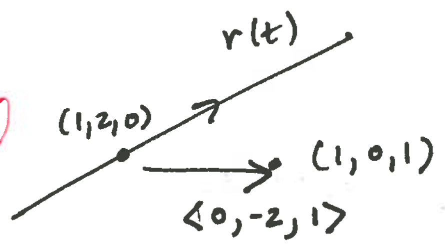
   $$= \hat{\imath} + \hat{\jmath} + 2\hat{k}$$

2. (10pts) Find an equation for the plane which contains the point $(1, 0, 1)$ and the line parameterized by

   $$\mathbf{r}(t) = (t+1)\mathbf{i} + (2-t)\mathbf{j} + t\mathbf{k}.$$

   $$= (\hat{\imath} + 2\hat{\jmath}) + t(\hat{\imath} - \hat{\jmath} + \hat{k})$$

   10

   $$\bar{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = \hat{\imath}(-1+2) + \hat{\jmath}(-1) + \hat{k}(-2)$$

   $$= \hat{\imath} - \hat{\jmath} - 2\hat{k}.$$

   $$x - y - 2z = -1$$

26

3. (25pts) Consider the function $f(x, y) = x^2y - y$.

(a) (4pts) Find ∇f, the gradient of f.

4

$$\nabla f = \langle 2xy, x^2 - 1 \rangle$$

(b) (6pts) Find the equation of the tangent plane to the graph of f at the point $(2, 1)$.

$\nabla f(2,1) = \langle 4, 3 \rangle$

6

$f(2,1) = 3$

$$z - 3 = 4(x-2) + 3(y-1)$$

(c) (5pts) Find the critical points of f.

$\langle 0, 0 \rangle = \langle 2xy, x^2-1 \rangle$

5

$0 = 2xy$

$0 = x^2 - 1 \Rightarrow x = \pm 1 \Rightarrow y = 0.$

cps: $(1,0)$, $(-1,0)$

(d) (10pts) Find the maximum and minimum **values** attained by the function on the disk $x^2+y^2 \leq 4$. **Note:** Check the values of the function at the critical points in the interior of the disk $x^2+y^2 < 4$, then find the maximum and minimum on the boundary of the disk $x^2+y^2 = 4$. For the boundary, you can use the technique of Lagrange multipliers, but other methods will work too.

On bondary, $x^2+y^2 = 4 \Rightarrow x^2 = 4 - y^2$.

10

So $f(x,y) = x^2y - y = (4-y^2)y - y = 3y - y^3 = f(y)$

Find max/min of $f(y)$ on $[-2, 2]$.

$f'(y) = 3 - 3y^2 \Rightarrow y = \pm 1.$

$f(1) = 2$, $f(-1) = -2$, $f(2) = -2$, $f(-2) = 2$.

So max/min valves on bondary are ± 2.

Check cps in interior:

$f(1,0) = 0$

$f(-1,0) = 0.$

max valve $= 2$

min valve $= -2$

or Lagrange Multipliers

$g(x,y) = x^2+y^2-4$

$\nabla f = \lambda \nabla g$

$g = 0$

$2xy = 2\lambda x$

$x^2 - 1 = 2\lambda y$

$x^2+y^2-4 = 0$

$\Longrightarrow$ $x = 0$ or $y = \lambda$

$x = 0 \Downarrow y = \pm 2$: $(0,2)$, $(0,-2)$, $(\pm 3, \pm 1)$

$y = \lambda \Downarrow$ $x^2 - 1 = 2y^2$, $x^2 - 2y^2 = 1$, $x^2 + y^2 = 4$ $\Rightarrow y^2 = \pm 1$, $x^2 = \pm 3.$

25

4. (12pts) Evaluate the following double integrals:

(a) (6pts) $\iint_R (2xy + 4y)\, dA$, where R is the rectangle $0 \le x \le 1, 0 \le y \le 2$.

6

$$= \int_0^1\int_0^2 (2xy+4y)\,dy\,dx = \int_0^1 \left(xy^2+2y^2\Big|_0^2\right. dx = \int_0^1 (4x+8)\,dx$$

$$= \left(2x^2+8x\Big|_0^1\right. = 10$$

(b) (6pts) $\iint_S (x^2 - 2x + y^2)\, dA$, where S is the unit disk $x^2 + y^2 \le 1$.

Switch to polar coords.

6

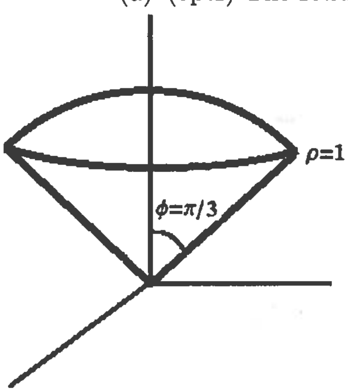
$$= \int_0^{2\pi}\int_0^1 (r^2-2r\cos\theta)\,r\,dr\,d\theta = \int_0^{2\pi} \left(\frac{r^4}{4}-\frac{2}{3}r^3\cos\theta\Big|_0^1\right. d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}-\frac{2}{3}\cos\theta\right) d\theta = \frac{\pi}{2}$$

5. (16pts) Use triple integrals in cylindrical or spherical coordinates to compute the volumes of the following solids. Remember,

$$V(S) = \iiint_S 1\, dV$$

(a) (8pts) The solid 'snocone' shape bounded by the sphere of radius one and the cone $\phi = \frac{\pi}{3}$.



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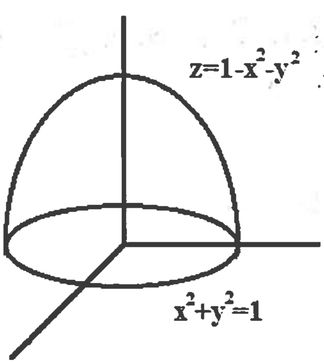
$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \left(\int_0^{\pi/3} \sin\phi\, d\phi\right)\left(\int_0^1 \rho^2\, d\rho\right)$$

$$= 2\pi \left(-\cos\phi\Big|_0^{\pi/3}\right.\left(\frac{1}{3}\rho^3\Big|_0^1\right. = 2\pi\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{\pi}{3}$$

(b) (8pts) The solid 'gumdrop' shape given by $0 \le z \le 1 - x^2 - y^2$.



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$$V = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^1 (r-r^3)\, dr$$

$$= 2\pi\left(\frac{r^2}{2}-\frac{r^4}{4}\Big|_0^1\right. = \frac{\pi}{2}$$

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6. (8pts) Match the following vector fields with the descriptions of the level surfaces to which they are perpendicular by writing the letter in the blank provided.

8

B $\mathbf{F}(x, y, z) = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$

D $\mathbf{F}(x, y, z) = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

A $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

C $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$

A. spheres

B. ellipsoids

C. paraboloids

D. planes

7. (17pts) Consider the vector field $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x + y)\mathbf{j}$.

(a) (6pts) Compute curl $\mathbf{F}$. **Note:** Consider $\mathbf{F}$ as a vector field in three-space with zero $\mathbf{k}$ component.

6

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x-y & x+y & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 2\hat{k}$$

$$= 2\hat{k}$$

3

(b) (3pts) Is $\mathbf{F}$ a gradient (conservative) vector field? Circle one: YES NO

(c) (8pts) Compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the unit circle centered at the origin in the xy-plane, traversed counterclockwise.

$$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} \qquad 0 \le t \le 2\pi$$

$$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

8

$$\int_C F \cdot dr = \int_0^{2\pi} \left((\cos t - \sin t)\hat{i} + (\cos t + \sin t)\hat{j}\right) \cdot \left(-\sin t\,\hat{i} + \cos t\,\hat{j}\right) dt$$

$$= \int_0^{2\pi} -\sin t\cos t + \sin^2 t + \cos^2 t + \sin t \cos t \, dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = 2\pi$$

25

8. (16pts) Let $\mathbf{F}(x,y) = (x^2 - y)\mathbf{i} + (x^3 + y)\mathbf{j}$ and let C denote the square with vertices $(0,0), (1,0), (1,1)$, and $(0,1)$ traversed counterclockwise.

(a) (8pts) Compute the line integral $\int_C \mathbf{F} \cdot dr$ directly or by using Green's Theorem.

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y & x^3+y & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (3x^2+1)\hat{k}$$

8

$$\int_C F \cdot dr \overset{\text{Green's}}{=} \int_0^1 \int_0^1 (3x^2+1)\, dx\, dy = (x^3 + x \Big|_0^1 = 2$$

(b) (8pts) Compute the line integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$ directly or by using the plane Divergence Theorem.

$$\text{div } F = 2x + 1.$$

8

$$\int_C F \cdot n\, ds \overset{\text{Div}}{=} \int_0^1 \int_0^1 (2x+1)\, dx\, dy = (x^2 + x \Big|_0^1 = 2$$

9. (8pts) Consider the vector field $\mathbf{F}(x,y,z) = xy^2\mathbf{i} + (yz^2 + x)\mathbf{j} + (zx^2 + xy)\mathbf{k}$. Use the Divergence Theorem to compute

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where S is the surface of the unit sphere centered at the origin and $\mathbf{n}$ is the outward pointing unit normal vector.

8

$$\text{div } F = y^2 + z^2 + x^2 = \rho^2.$$

Using spherical coords

$$\iint_S F \cdot n\, dS = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \left(\int_0^{\pi} \sin\phi \, d\phi \right) \left(\int_0^1 \rho^4 \, d\rho \right)$$

$$= \frac{4\pi}{5}$$

10. (6pts) Suppose **F** is a gradient (conservative) vector field. In other words, there is a potential function $f(x, y)$ such that $\mathbf{F}(x, y) = \nabla f(x, y)$. Suppose C is unit circle centered at the origin in the xy-plane, traversed counterclockwise. Which **two** quantities below must be equal to zero (or the zero vector)? Circle **two** letters.

6

A. $\int_C \mathbf{F} \cdot d\mathbf{r}$ B. $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$ C. $\operatorname{curl} \mathbf{F}$ D. $\operatorname{div} \mathbf{F}$

11. (16pts) Let S denote the surface determined by

$$z = \frac{1}{2}x^2 + \frac{1}{2}y^2,$$

where $x^2 + y^2 \leq 1$ (S is the graph of $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ over the unit circle).

(a) (8pts) Find the surface area of S.

$$\frac{\partial z}{\partial x} = x \qquad \frac{\partial z}{\partial y} = y.$$

$u = 1 + r^2$
$du = 2r\,dr$

8

$$SA = \iint_{\text{circle}} \sqrt{1 + x^2 + y^2}\, dA = \int_0^{2\pi} \int_0^1 r\sqrt{1+r^2}\, dr\, d\theta$$

$$= 2\pi \left(\frac{1}{2}\right) \int_1^2 u^{1/2}\, du = \pi \left(\frac{2}{3}\right)\left(2^{3/2} - 1\right)$$

$$= \frac{2\pi}{3}\left(\sqrt{8} - 1\right)$$

(b) (8pts) Use Stokes's Theorem to evaluate

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where **F** denotes the vector field

$$\mathbf{F} = (1 - 2z + x)\mathbf{i} + 4z^2 y\mathbf{j} + (x^2 - y^2)\mathbf{k}.$$

8

$r(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + \tfrac{1}{2}\hat{k} \quad 0 \leq t \leq 2\pi$

$r'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 0\hat{k}.$

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS \underset{\text{Stokes}}{=} \int_0^{2\pi} \left((\cos t)\hat{\imath} + \sin t\,\hat{\jmath} + (\cos^2 t - \sin^2 t)\hat{k}\right) \cdot \left(-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}\right) dt$$

$$= \int_0^{2\pi} (\sin t + \cos t \sin t \cos t)\, dt$$

$$= 0$$