2210-90 Final Exam
Fall 2012

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Please try to do all all work in the space provided. Please circle your final answer.

1. (15pts) Let $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j} - \mathbf{k}$.

   (a) (3pts) Find $\mathbf{u} - \mathbf{v}$

   $$2\hat{i} - 3\hat{j} + \hat{k} - (3\hat{i} + 5\hat{j} - \hat{k}) = -\hat{i} - 8\hat{j} + 2\hat{k}$$

   (b) (4pts) Find $||\mathbf{u}||$

   $$||u|| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

   (c) (4pts) Find $\mathbf{u} \cdot \mathbf{v}$

   $$u \cdot v = (2\hat{i} - 3\hat{j} + \hat{k}) \cdot (3\hat{i} + 5\hat{j} - \hat{k}) = 6 - 15 - 1 = -10$$

   (d) (4pts) Find $\mathbf{u} \times \mathbf{v}$

   $$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 1 \\ 3 & 5 & -1 \end{vmatrix} = -2\hat{i} + 5\hat{j} + 19\hat{k}$$

2. (10pts) Find the equation for the plane which contains the points $(1, 0, 1)$, $(3, -1, -1)$, and $(2, 5, 0)$.

   $$\langle 3,-1,-1\rangle - \langle 1,0,1\rangle = \langle 2,-1,-2\rangle = u$$
   $$\langle 2,5,0\rangle - \langle 1,0,1\rangle = \langle 1,5,-1\rangle = v$$

   4

   $$n = u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ 1 & 5 & -1 \end{vmatrix} = 11\hat{i} + 11\hat{k}.$$

   Plug in a point, say $(1,0,1)$

   $$11x + 11z = 22$$

   or

   $$x + z = 2$$

3. (15pts) Consider the function $f(x, y) = x\sqrt{y}$.

(a) (5pts) Find ∇f, the gradient of f.

$$\nabla f(x,y) = \left\langle \sqrt{y}, \frac{x}{2\sqrt{y}} \right\rangle$$

(b) (5pts) Find the equation of the tangent plane to the graph of f at the point $(3, 4)$.

$$f(3,4) = 3\sqrt{4} = 6.$$

$$z - 6 = 2(x-3) + 3/4\,(y-4)$$

(c) (5pts) Use part (b) above to estimate the value of $(3 + \frac{1}{3})\sqrt{4 + \frac{2}{3}}$.

$$\approx 6 + 2\left(3+\tfrac{1}{3}-3\right) + \tfrac{3}{4}\left(4+\tfrac{2}{3}-4\right)$$

$$= 6 + \tfrac{2}{3} + \tfrac{1}{2} = \tfrac{36}{6} + \tfrac{7}{6} = 43/6$$

4. (12pts) Use Lagrange multipliers to find the extreme values (both maximum and minimum) of the function $f(x, y) = xy$ on the ellipse $x^2 + 4y^2 = 1$. **Hint:** Solve for λ, then use the constraint equation.

$$g(x,y) = x^2 + 4y^2 - 1$$

$$\nabla f(x,y) = \lambda \nabla g(x,y) \Rightarrow \langle y, x\rangle = \lambda \langle 2x, 8y\rangle$$

① $y = 2\lambda x$

② $x = 8\lambda y$

③ $x^2 + 4y^2 = 1$

5

① $\Rightarrow \lambda = y/2x$

② $\Rightarrow \lambda = x/8y$

$\Rightarrow y/2x = x/8y \Rightarrow 8y^2 = 2x^2 \Rightarrow 4y^2 = x^2.$

Now use ③ to conclude

$x^2 = 4y^2 = \frac{1}{2}$

This gives the points

$(1/\sqrt{2}, 1/\sqrt{8})$ $(-1/\sqrt{2}, 1/\sqrt{8})$

$(1/\sqrt{2}, -1/\sqrt{8})$ $(-1/\sqrt{2}, -1/\sqrt{8})$

3

Now plug in to find

max $= 1/4$

min $= -1/4$

2

5. Evaluate the following double integrals:

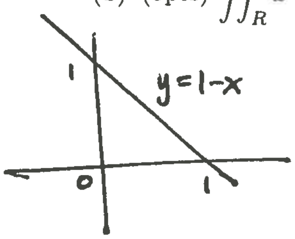
(a) (5pts) $\iint_R (x^2+2y)\,dA$, where R is the rectangle $0 \le x \le 1, 2 \le y \le 3$.

$$\underbrace{\int_2^3\int_0^1 (x^2+2y)\,dx\,dy}_{2} = \int_2^3 \left(\frac{x^3}{3}+2xy\Big|_0^1\right) dy$$

$$= \int_2^3 \left(\frac{1}{3}+2y\right) dy$$

$$= \left(\frac{1}{3}y+y^2\Big|_2^3 = 1+9-\frac{2}{3}-4 = \frac{16}{3}\right.$$

(b) (5pts) $\iint_R x\,dA$, where R is the region bounded by the x-axis, the y-axis, and the line $y = 1-x$.



$$= \underbrace{\int_0^1\int_0^{1-x} x\,dy\,dx}_{3} = \int_0^1 (x-x^2)\,dx$$

$$= \left(\frac{x^2}{2}-\frac{x^3}{3}\Big|_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}\right.$$

or

$$\int_0^1\int_0^{1-y} x\,dx\,dy$$

(c) (5pts) $\iint_R \sin(x^2+y^2)\,dA$, where R is the region $x^2+y^2 \le \pi$.

Switch to polar coordinates 2

$$= \underbrace{\int_0^{2\pi}\int_0^{\sqrt{\pi}} \sin(r^2)\,r\,dr\,d\theta}_{2} = 2\pi\int_0^{\sqrt{\pi}} \sin(r^2)\,r\,dr$$

$u = r^2$

$du = 2r\,dr$

$$= \pi\int_0^{\pi} \sin(u)\,du$$

$$= \pi\left(-\cos u\Big|_0^{\pi}\right.$$

$$= \pi(1+1) = 2\pi$$

6. Evaluate the following triple integrals:

(a) (5pts) $\iiint_R (x^3z + y)\, dV$, where R is the rectangle $0 \le x \le 1, 0 \le y \le 1$, and $1 \le z \le 3$.

$$\int_1^3\int_0^1\int_0^1 (x^3z+y)\,dx\,dy\,dz = \int_1^3\int_0^1 \left(\frac{x^4}{4}z + yx\right|_0^1 dy\,dz$$

$$= \int_1^3\int_0^1 \left(\frac{1}{4}z + y\right) dy\,dz$$

$$= \int_1^3 \left(\frac{1}{4}zy + \frac{y^2}{2}\right|_0^1 dz$$

$$= \int_1^3 \left(\frac{1}{4}z + \frac{1}{2}\right) dz = \left(\frac{z^2}{8} + \frac{1}{2}z\right|_1^3 = \frac{9}{8} + \frac{3}{2} - \frac{1}{8} - \frac{1}{2}$$

$$= \frac{8}{8} + \frac{2}{2} = 2$$

(b) (5pts) $\iiint_R \frac{z}{\sqrt{x^2+y^2}}\, dV$, where R is the cylinder $x^2 + y^2 \le 1$, $0 \le z \le 2$.

Use cylindrical coordinates 2

$$= \int_0^{2\pi}\int_0^1\int_0^2 \frac{z}{r}\, r\,dz\,dr\,d\theta = 2\pi \int_0^2 z\,dz$$

2

$$= 2\pi\left(\frac{z^2}{2}\right|_0^2 = 4\pi$$

(c) (5pts) $\iiint_R z\, dV$, where R is the hemisphere $x^2 + y^2 + z^2 \le 1$, $z \ge 0$.

Use spherical coordinates 2

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\left(\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\right)\left(\int_0^1 \rho^3\,d\rho\right)$$

2

$$= \frac{\pi}{2}\left(\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\right)$$

$u = \sin\phi$
$du = \cos\phi\,d\phi$

$$= \frac{\pi}{2}\int_0^1 u\,du$$

$$= \frac{\pi}{2}\left(\frac{u^2}{2}\right|_0^1 = \frac{\pi}{4}$$

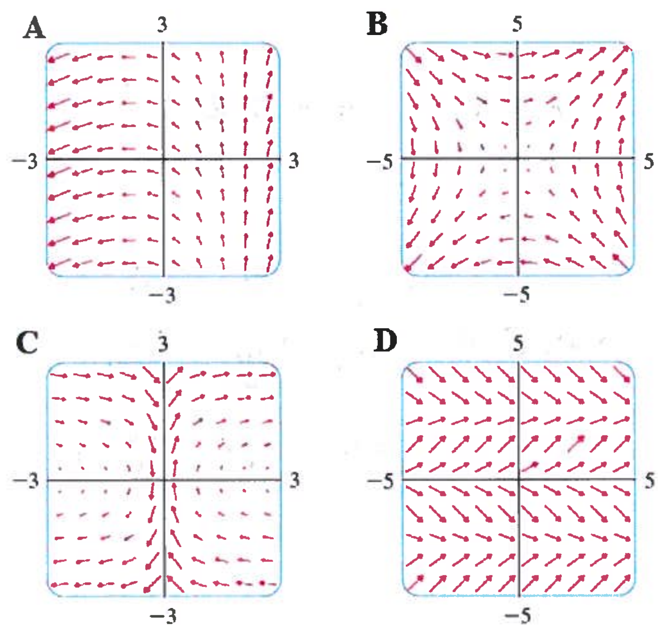
7. (8pts) Match the equation of the vector field with its plot by writing the letter in the blank provided.

B $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$

D $\mathbf{F}(x, y) = \mathbf{i} + \sin y\mathbf{j}$

A $\mathbf{F}(x, y) = (x-2)\mathbf{i} + (x+1)\mathbf{j}$

C $\mathbf{F}(x, y) = y\mathbf{i} + \frac{1}{x}\mathbf{j}$



8. (12pts) Consider the vector field $\mathbf{F}(x, y) = (2x - 2y)\mathbf{i} + (4y - 2x)\mathbf{j}$.

(a) (6pts) Find a function $f(x, y)$ such that $\nabla f = \mathbf{F}$.

$$\frac{\partial f}{\partial x} = 2x - 2y \Rightarrow f = x^2 - 2xy + c_1(y)$$

$$\frac{\partial f}{\partial y} = 4y - 2x \Rightarrow f = 2y^2 - 2xy + c_2(x)$$

$$f(x,y) = x^2 + 2y^2 - 2xy$$

(b) (6pts) Let C denote the curve with the parameterization

$$\mathbf{r}(t) = (1 + 2t)\mathbf{i} + (e^t \sin(2\pi t) - t^2)\mathbf{j}$$

for $0 \leq t \leq 1$. Use part (a) above to find

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

$r(0) = (1, 0)$

$r(1) = (3, -1)$

$$\int_C F \cdot dr = f(3,-1) - f(1,0) = 17 - 1 = 16$$

9. (10pts) Consider the vector field $\mathbf{F}(x, y) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$.

(a) (4pts) Find div $\mathbf{F}$

$$\text{div } F = z + xz$$

(b) (4pts) Find curl $\mathbf{F}$

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xyz & -y^2 \end{vmatrix} = (-2y - xy)\hat{i} + x\hat{j} + yz\hat{k}$$

(c) (2pts) Is $\mathbf{F}$ a conservative vector field? Circle one: YES NO

10. (18pts) Let $\mathbf{F}(x, y) = (-2y + x)\mathbf{i} + (2x)\mathbf{j}$, and let C denote the counter-clockwise unit circle with parameterization

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$$

for $0 \leq t \leq 2\pi$.

(a) (6pts) Compute directly the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

$$\int_C F \cdot dr = \int_0^{2\pi} \left((-2\sin t + \cos t)\hat{i} + (2\cos t)\hat{j} \right) \cdot (-\sin t\,\hat{i} + \cos t\,\hat{j})\, dt$$

$$= \int_0^{2\pi} 2\sin^2 t - \sin t \cos t + 2\cos^2 t\, dt = \int_0^{2\pi} (2 - \cos t \sin t)\, dt = 4\pi$$

(b) (6pts) Use Green's Theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ -2y+x & 2x \end{vmatrix} = 2 + 2 = 4.$$

2

$$\int_C F \cdot dr = \iint_S 4\, dA = 4\pi$$

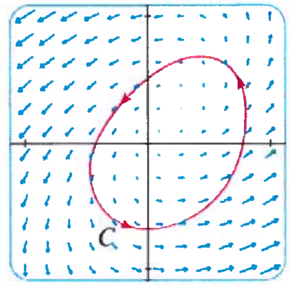
(c) (6pts) Use the Divergence Theorem (plane version) to compute $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$.

$$\text{div } F = 1.$$

$$\int_C F \cdot n\, ds = \iint_S 1\, dA = \pi$$

2

11. (6pts) Indicate whether each statement is true (T) or false (F) by writing in the blank provided.



T $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$

F $\int_C \mathbf{F} \cdot \mathbf{n}\, ds < 0$

F $\mathbf{F}$ is conservative.

12. (14pts) Let S denote the surface determined by

$$z = 1 - x^2 - y^2,$$

where $x^2 + y^2 \leq 1$ (S is the graph of $f(x, y) = 1 - x^2 - y^2$ over the unit circle).

(a) (7pts) Find the surface area of S.

$$\frac{\partial z}{\partial x} = -2x \qquad \frac{\partial z}{\partial y} = -2y \qquad \text{polar}$$

$$SA = \iint_S \sqrt{1+(2x)^2+(2y)^2}\, dA = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\, dr\, d\theta$$

4

$$= 2\pi \int_0^1 \sqrt{1+4r^2}\, r\, dr\, d\theta = \frac{\pi}{4}\int_1^5 \sqrt{u}\, du = \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^5$$

$u = 1+4r^2$

$du = 8r\,dr$

$$= \frac{\pi}{6}\left(5^{3/2}-1\right)$$

(b) (7pts) Use Stokes Theorem to evaluate

$$\iint_S (\text{curl}\, \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where $\mathbf{F}$ denotes the vector field

$$\mathbf{F} = y\mathbf{i} - x\mathbf{j} + xy\mathbf{k}.$$

$$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\hat{k}$$

$$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}$$

$$\iint_S (\text{curl}F)\cdot n\, dS = \int_C F\cdot dr = \int_0^{2\pi} (\sin t\,\hat{i} - \cos t\,\hat{j} + \cos t \sin t\,\hat{k})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j})\, dt$$

4

$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\, dt = -2\pi$$

± ok