CHAPTER 9

Sequences and Series

§9.1. Convergence: Definition and Examples

Sequences

The purpose of this chapter is to introduce a particular way of generating algorithms for finding the values of functions defined by their properties; for example, transcendental functions. This is the technique of *Infinite Series*. Computer algorithms for determining the value of a function depend upon the usual arithmetic operations; thus an exact determination can only be achieved for rational functions (quotients of polynomials). If a function is transcendental, its values can only be approximated. For example, we know that

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n . \tag{9.1}$$

This expression tells us that if for any n we do the calculation described by the expression on the right, that these numbers will, for n large enough, be close to the "true" value of e^x. Now, it turns out that this is a very inefficient way to calculate e^x, and the expression as an infinite series (which we will discover later in this chapter)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \tag{9.2}$$

is far better. Equation 9.2 is taken to mean: add up the numbers of the form $x^n/n!$, starting with $n = 0$. If we add up enough terms, we have a good approximation to e^x. Of course, it is important to have estimates on how good this approximation is, and more generally, to have ways of finding these approximating sums. That is what we study in this chapter, starting with the idea of convergence in the sense of "good approximation".

Definition 9.1 *A sequence is a list of numbers, denoted $\{a_n\}$, where a_n is the nth term of the sequence.*

A sequence may be defined by a specific formula or an algorithm for determining the members of the sequence successively, or *recursively*.

Example 9.1 The formulae $a_n = n \ ,n \geq 1$; $b_n = \dfrac{n+1}{n-1} \ ,n \geq 2$; $c_n = 3+2n, n \geq 0$ define the sequences, respectively:

$$1,2,3,\ldots,n,\ldots \ ; \qquad \frac{3}{1},\frac{4}{2},\frac{5}{3},\ldots,\frac{n+1}{n-1},\ldots \ ; \qquad 3,5,7,9,\ldots,3+2n,\ldots \tag{9.3}$$

The last sequence can be defined recursively by: $a_0 = 3$, and for $n > 0$, $c_n = c_{n-1} + 2$. Similarly, the first is given by the recursion $a_1 = 1, \ a_n = a_{n-1} + 1$.

The symbol $n!$ (read "n-factorial") is used to denote the product of the first n integers. This also has the recursive definition: $a_0 = 1$, and for $n > 0$, $a_n = na_{n-1}$. (Note that we have defined $0! = 1$).

Of the sequences described in equation 9.2, the first and the third clearly grow without bound, but the second is bounded; in fact, if we rewrite the general term as

$$b_n = \frac{n+1}{n-1} = \frac{1+\frac{1}{n}}{1-\frac{1}{n}} \ , \tag{9.4}$$

we see that the sequence b_n approaches 1. We say that b_n converges to 1, as in the following definition.

Definition 9.2 *A sequence* $\{a_1, a_2, \ldots, a_n, \ldots\}$ *converges to a limit L, written*

$$\lim_{n\to\infty} a_n = L \ , \tag{9.5}$$

if, for every $\varepsilon > 0$*, there is an* n_0 *such that for all* $n \geq n_0$ *we have* $|a_n - L| < \varepsilon$.

This just says that we can be sure that a_n is as close to L as we need it to be, just by taking the index n large enough. We will rarely have to actually use this definition, relying more on understanding what it says, and known facts about limits. For example:

Proposition 9.1 *If the general term* a_n *of a sequence can be expressed as* $f(n)$ *for a continuous function* f *and if we know that* $\lim_{x\to\infty} f(x) = L$*, then we can conclude that* $\lim_{x\to\infty} a_n = L$.

As an application, using results from the preceding chapter, we have

Proposition 9.2

a) $\displaystyle\lim_{n\to\infty} n^p = \infty$ for $p > 0$,

b) $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$,

c) $\displaystyle\lim_{n\to\infty} A^{1/n} = 1$ if $A > 0$.

Let p and q be polynomials.

d) $\displaystyle\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0$ if $\deg p < \deg q$, $\displaystyle\lim_{n\to\infty} \frac{p(n)}{q(n)} = \infty$ if $\deg p > \deg q$.

e) If the polynomials p and q have the same degree, then $\displaystyle\lim_{n\to\infty} \frac{p(n)}{q(n)} = \frac{a}{b}$ *, where a and b are the leading coefficients of p and q.*

f) $\displaystyle\lim_{n\to\infty} \frac{p(n)}{e^n} = 0$ *for any polynomial p.*

g) $\displaystyle\lim_{n\to\infty} \frac{p(n)}{\ln(n)^c} = \infty$ *for any polynomial of positive degree and any positive c.*

These can all be derived by replacing n by x, and using limit theorems already discussed (such as l'Hôpital's rule).

Example 9.2 $\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1$, by (e) above .

Example 9.3 $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$, since the numerator oscillates between -1 and 1, and the denominator goes to infinity. We should not be perturbed by such oscillation, so long as it remains bounded. For example we also have

$$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0 \ , \tag{9.6}$$

since the term $\sin(n)$ remains bounded. The following propositions state the general rule for handling such cases.

Proposition 9.3

a) (Squeeze theorem) Given three sequences $a_n,\ b_n, c_n$, *if* $a_n \geq b_n \geq c_n$ for all n , and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, *then also* $\lim_{n\to\infty} b_n = L$.

b) If $a_n = b_n c_n$, *the sequence* b_n *is bounded,* $c_n \geq 0$ *and* $\lim_{n\to\infty} c_n = 0$, *then also* $\lim_{n\to\infty} a_n = 0$.

Let's see why b) is true, using a). Let M be the bound of the $|b_n|$. Then

$$Mc_n \geq b_n c_n \geq -Mc_n \tag{9.7}$$

so a) applies and the conclusion follows.

In some cases where none of the above rules apply, we have to return to the definition of convergence.

Example 9.4 For any $a > 0$, $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.

To see why this is true, we think of the sequence as recursively defined: the first term, a_1 is a, and each a_n is obtained by multiplying its predecessor by a/n. Now, eventually, that is, for n large enough, $a/n < 1/2$. Thus each term after that is less than half its predecessor. This now surely looks like a sequence converging to zero. To be more precise, let N be the first integer for which $a/N < 1/2$. Then for any $k > 0$,

$$\frac{a^{N+k}}{(N+k)!} < \frac{1}{2^k}\frac{a^N}{N!} \ . \tag{9.8}$$

Now the sequence on the right is a fixed number (a^N/N) times a sequence ($1/2^k$) which tends to zero. Thus our sequence converges to zero, also by the squeeze theorem (proposition 9.3a).

Note that in the above argument, we only had to show that the general term of our sequence is dominated by the general term of a sequence converging to zero *from some point on*. What happens to any finite collection of terms of a sequence is not relevant to the question of convergence. We shall use the word *eventually* to mean "from some point on", or more precisely, "for all n greater than some fixed integer N". We restate proposition 9.3, using the word "eventually":

Proposition 9.4

a) (Squeeze theorem) Given three sequences a_n, b_n, c_n, if eventually

$$a_n \geq b_n \geq c_n \text{ for all n , and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \ , \tag{9.9}$$

then also

$$\lim_{n\to\infty} b_n = L \ . \tag{9.10}$$

b) Suppose that $a_n = b_n c_n$ eventually, that is, for all n larger than some N. If the sequence b_n is bounded and

$$\lim_{n\to\infty} c_n = 0, \tag{9.11}$$

then also

$$\lim_{n\to\infty} a_n = 0. \tag{9.12}$$

Example 9.5 For any positive integer p, $\lim_{n\to\infty} \dfrac{n^p}{n!} = 0$.

The idea here is that the numerator is a product of p terms, whereas the denominator is a product of n terms, so grows faster than the numerator. To make this precise, write

$$\frac{n^p}{n!} = \frac{n \cdots n}{n(n-1)\cdots(n-p+1)} \frac{1}{(n-p)!} \ . \tag{9.13}$$

Now, if n is so large that $n/(n-p) < 2$, ($n > 2p$ will do), then the first factor is bounded by 2^p. Thus, for $n > 2p$, that is, eventually,

$$\frac{n^p}{n!} < 2^p \frac{1}{(n-p)!} \ . \tag{9.14}$$

Since $1/(n-p)! \to 0$ and $n \to \infty$, the result follows from the squeeze theorem.

Finally, we note that the limit of a sum is the sum of the limits:

Proposition 9.5 *If $a_n = b_n + c_n$, and the sequences b_n and c_n converge, then so does the sequence a_n, and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n + \lim_{n\to\infty} c_n \ . \tag{9.15}$$

Series

For many sequences, in fact, the most important ones, the general term is formed by adding something to its predecessor; that is, the sequence is formed by the recursion $s_n = s_{n-1} + a_n$, where a_n is from another sequence. Such a sequence is called a *series*. Explicitly, the terms of the series are

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + a_3 + \cdots + a_n, \cdots \ . \tag{9.16}$$

It is useful to use the summation symbol:

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \ . \tag{9.17}$$

Definition 9.3 *The series*

$$\sum_{k=0}^{\infty} a_k \tag{9.18}$$

is to be considered as the limit of the sequence

$$s_n = \sum_{k=0}^{n} a_k \ . \tag{9.19}$$

If the limit L of the sequence $\{s_n\}$ exists, the series is said to converge, and L is called its sum. If the limit does not exist, the series diverges. The terms of the sequence $\{s_n\}$ are called the partial sums of the series.

Example 9.6 $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

Let's look at a few partial sums:

$$\frac{1}{2}\ ,\ \frac{3}{4}\ ,\ \frac{7}{8}\ ,\ \frac{15}{16}\ ,\ \ldots \tag{9.20}$$

We see that each term adds half the distance of its predecessor from 1, from which we guess that the partial sum: $s_n = 1 - 2^{-n}$. Let's now show that to be true. As we have seen it is true for the first four terms. If it is true for the $(n-1)$th term, it is also true for the nth term:

$$s_n = s_{n-1} + \frac{1}{2^n} = 1 - \frac{1}{2^{n-1}} + \frac{1}{2^n} = 1 - \frac{2-1}{2^n} = 1 - \frac{1}{2^n} \ . \tag{9.21}$$

Thus, our guess holds for the fifth, and then the sixth, and, by continued application of equation 9.21, ultimately, for all terms. So the result is easy to conclude:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} (1 - \frac{1}{2^n}) = 1 \ . \tag{9.22}$$

Now, remember that the index is a way of relating the partial sums of the series to the general term from which it is defined, so if we change that relation consistently, we don't change the series. For example,

$$\sum_{k=1}^{\infty} a_k = \sum_{n=1}^{\infty} a_n = \sum_{k=0}^{\infty} a_{k+1} = \sum_{m=9}^{\infty} a_{m-8} \tag{9.23}$$

and so forth. Each representation comes about by replacing the index with a new index. For example, if we substitute n for k, we get the first equality; if we substitute $k+1$ for n we get the second equality, and if we replace $k+1$ by $m-8$, we get the last one. It is often useful to make a change of index as the next examples show.

Example 9.7 $\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$.

For

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 + 1 = 2 . \tag{9.24}$$

Example 9.8 $\sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}$.

First, change the index by $k = m+n$, and then factor out 2^{-n}:

$$\sum_{k=n}^{\infty} \frac{1}{2^k} = \sum_{m=0}^{\infty} \frac{1}{2^{m+n}} = 2^{-n} \sum_{m=0}^{\infty} \frac{1}{2^m} = 2^{-n} \cdot 2 = 2^{-n+1} . \tag{9.25}$$

Proposition 9.6 *(Geometric Series) :*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for } |x| < 1 , \tag{9.26}$$

$$\sum_{k=0}^{\infty} x^k \quad \text{diverges for } |x| \geq 1 . \tag{9.27}$$

To show this, we obtain (by a clever little observation) a formula for the partial sums

$$s_n = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n . \tag{9.28}$$

Note that

$$s_{n+1} = (1 + x + x^2 + \cdots + x^n) + x^{n+1} = s_n + x^{n+1} \quad \text{and} \tag{9.29}$$

$$s_{n+1} = 1 + (x + x^2 + \cdots + x^{n+1}) = 1 + x s_n . \tag{9.30}$$

Equating these expressions for s_{n+1}, we obtain $s_n + x^{n+1} = 1 + x s_n$. Solving this for s_n:

$$s_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1-x} , \tag{9.31}$$

so

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1-x} , \tag{9.32}$$

which equals $(1-x)^{-1}$ if $|x|<1$ and diverges if $|x|>1$. We look at the cases $x=\pm 1$ separately. For $x=1$, $s_n=n$, so the series diverges. For $x=-1$, the sequence s_n is the sequence $1,0,1,0,1,0,\ldots$, so cannot converge to any particular number.

Example 9.9 $\sum_{n=1}^{\infty}\frac{1}{k(k+1)}=1$. We first use the fact that

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}\,. \tag{9.33}$$

Thus the partial sum s_n can be calculated:

$$s_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) \tag{9.34}$$

$$=1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)-\cdots+\left(-\frac{1}{n}+\frac{1}{n}\right)-\frac{1}{n+1} \tag{9.35}$$

$$=1-\frac{1}{n+1}\,, \tag{9.36}$$

which converges to 1 as n goes to infinity. This is an example of a *telescoping series*.

Finally, we observe that if a series converges, its general term must go to zero. **Be careful: there are many series whose general term goes to zero which do not converge.**

Proposition 9.7 If $\sum_{\mathrm{k=0}}^{\infty}\mathrm{a_k}$ *converges, then* $\lim_{n\to\infty}a_k=0$.

To see this, let $s_n=\sum_{k=0}^{n}a_k,\ t_n=\sum_{k=0}^{n-1}a_k$. Then, since these are both sequences of the partial sums of the series, but indexed differently, $\lim_{n\to\infty}s_n=\lim_{n\to\infty}t_n$. Thus $\lim_{n\to\infty}(s_n-t_n)=0$. But $s_n-t_n=a_n$.

Finally, Proposition 9.5 gives us:

Proposition 9.8 *If $a_n=b_n+c_n$, and the series $\sum b_n$ and $\sum c_n$ converge, then so does the series $\sum a_n$, and $\sum a_n=\sum b_n+\sum c_n$.*

§9.2. Tests for convergence

Throughout this section, unless otherwise specified, we will be considering series, all of whose terms are positive. For such a series, the sequence of partial sums is increasing. If they remain bounded, then - just as in the assertion of theorem 8.1 for functions - the sequence of partial sums will converge.

Proposition 9.9 *If $a_k\geq 0$ for all k, and there is an $M>0$ such that $\sum_{k=0}^{n}a_k\leq M$ for all n, then $\sum_{k=0}^{\infty}a_k$ converges.*

Because of this proposition, for a series with positive terms, the statements $\sum a_k$ converges, $\sum a_k$ diverges, are usually written simply as

$$\sum_{k=0}^{\infty} a_k < \infty \ \text{(converges)}\ , \qquad \sum_{k=0}^{\infty} a_k = \infty \ \text{(diverges)}\ . \tag{9.37}$$

Here is an important application of this proposition:

Proposition 9.10 *(Comparison Test). Given two sequences a_k, b_k with $0 \le a_k \le b_k$. Then*
a) If $\sum b_k < \infty$, then $\sum a_k < \infty$,
b) If $\sum a_k = \infty$, then $\sum b_k = \infty$.

As for (a), the sequence of partial sums $s_n = \sum_0^n a_k$ is bounded by $\sum_0^\infty b_k$, so converges by Proposition 9.9. In the second case, since the sequence of partial sums $\sum a_k$ has no bound, neither does the sequence of partial sums of $\sum b_k$.

It is important to observe that it is not necessary that the inequalities in the hypothesis of proposition 9.10 hold for all k, only that they eventually hold. That is because the issue of convergence series is determined by the end of the series, and not affected by any finite number of terms.

Example 9.10 $\sum \frac{1}{r^k(r+1)} < \infty$ if $0 < r < 1$.

Since $r^{k+1} < r^k(r+1)$,

$$\frac{1}{r^k(r+1)} < \frac{1}{r^{k+1}}\ , \tag{9.38}$$

so the comparison test applies.

Example 9.11 $\sum \frac{k}{r^k} < \infty$ if $r > 1$.

Now, here the trouble is that the numerator grows without bound - but it doesn't grow as fast as a power. So, what we do is borrow something from the denominator to compensate for the numerator. We note that eventually $k/r^{k/2} < 1$; in fact, this is true as soon as $k > 2\ln k/\ln r$ (which eventually happens, since $k/\ln k \to \infty$). Then for all k larger than this number

$$\frac{k}{r^k} = \frac{k}{(\sqrt{r})^k}\frac{1}{(\sqrt{r})^k} < \frac{1}{(\sqrt{r})^k}\ . \tag{9.39}$$

Since $r > 1$, we also have $\sqrt{r} > 1$, and so the series

$$\sum \frac{1}{(\sqrt{r})^k} \tag{9.40}$$

converges, and thus, by comparison, our original series converges.

Example 9.12 $\sum_{n=0}^{\infty} \frac{1}{n^2} < \infty$.

Now,

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}\ , \tag{9.41}$$

so our series is dominated by a telescoping series which converges (see example 9.9. above).

A very useful application of the comparison test is the following.

Proposition 9.11 *(The Integral Test). Suppose that f is a nonnegative, nonincreasing function defined on an interval $[M,\infty)$. Suppose the a_n is a sequence such that for $n > M$, $a_n = f(n)$. Then*

a) If $\int_M^\infty f(x)dx < \infty$ then $\sum_{n=0}^\infty a_n < \infty$,

b) If $\int_M^\infty f(x)dx = \infty$ *then* $\sum_{n=0}^\infty a_n = \infty$.

Let

$$b_n = \int_n^{n+1} f(x)dx \; . \tag{9.42}$$

Then, since the function is nonincreasing, $f(n) \geq b_n \geq f(a_{n-1})$; that is $a_n \geq b_n \geq a_{n+1}$. Now, use the comparison theorem. For example, if $\int f(x)dx < \infty$, then $\sum b_n$ converges, so by comparison $\sum a_{n+1}$ also converges.

Example 9.13 (The harmonic series). $\displaystyle\sum_{n=1}^\infty \frac{1}{n} = \infty$.

We apply the integral test using the function $f(x) = 1/x$. Since

$$\int_1^\infty \frac{dx}{x} = \infty \; , \tag{9.43}$$

as we saw in chapter 8, the result follows.

If we apply example 17 of chapter 8 to series via the integral test we have a result which is very useful for comparisons:

Proposition 9.12 *Let p be a positive number.*

a) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p} < \infty$ *if* $p > 1$

b) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p} = \infty$ *if* $p \leq 1$.

This follows from the same result for the integral of $1/x^p$.

Example 9.14 $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$.

The function $f(x) = 1/x(\ln x)^p$ is decreasing. We integrate using the substitution $u = \ln x$:

$$\int_2^A \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\ln A} \frac{du}{u^p} \; . \tag{9.44}$$

We know (again from example 17, chapter 8) that this converges if $p > 1$, and otherwise diverges. Thus, by the integral test,

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^p} < \infty \quad \text{if p} > 1 \; , \tag{9.45}$$

and otherwise diverges.

Finally, we need a tool to test for convergence when we cannot realize the general term of the series in the form $f(n)$ for some function f. For example, if the expression for a_n involves the factorial, we proceed to the following.

Proposition 9.13 *(Ratio Test). Given the series $\sum a_n$, consider*

$$\lim \frac{a_{n+1}}{a_n} = L \, , \tag{9.46}$$

if the limit exists. If $L < 1$, the series converges; if $L > 1$, the series diverges. For the case $L = 1$, we can draw no conclusion.

Suppose that $L < 1$. Then there is a number r with $L < r < 1$ such that eventually $a_{n+1}/a_n < r$. That is, there is an integer N such that $a_{n+1}/a_n < r$ for all $n \geq N$. We conclude

$$a_{N+1} < a_N r \, , \quad a_{N+2} < a_{N+1} r < a_N r^2 \, , \quad a_{N+3} < a_{N+2} r < a_N r^3 \, , \tag{9.47}$$

and so forth. Thus, we have, for all $k \geq 1$, $a_{N+k} < a_N r^k$, so by comparison with the geometric series, our series converges.

If on the other hand, $L > 1$, there is a number r, $L > r > 1$, such that eventually $a_{n+1}/a_n > r$. Following the same argument but with the inequalities reversed, we conclude that for all $k \geq 1$, $a_{N+k}/a_N r^k$, so we have divergence by comparison with the geometric series. We can conclude nothing if $L = 1$. This is the case for the all the series of the type $\sum 1/n^p$, and as we have seen, for some p we get convergence, and divergence for other p.

Example 9.15 $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{n!}$.

We try the ratio test.

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{(n+1)!} \frac{n!}{a^n} = \frac{a}{n+1} \to 0 \tag{9.48}$$

as $n \to \infty$, so the ratio test gives us convergence.

Example 9.16 $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^3}{3^n}$.

Try the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)^3}{3^{n+1}} \frac{3^n}{2^n n^3} = \frac{2}{3} \left(\frac{n+1}{n} \right)^3 \to \frac{2}{3} \tag{9.49}$$

so we have convergence.

Example 9.17 $\displaystyle\sum_{n=1}^{\infty} r^n$.

Here the ratio test gives

$$\frac{a_{n+1}}{a_n} = r \, , \tag{9.50}$$

so we conclude that the series converges if $r < 1$, and diverges if $r > 1$. This may seem to be a simplification of proposition 9.6, but in fact it is a fraud. The argument is circular, for we have used the convergence of the geometric series to derive the ratio test.

We observe that we didn't really need to know that the limit of a_{n+1}/a_n exists, only that eventually these ratios are either less than some number less than 1 to conclude convergence, or greater than some number greater than 1, for divergence.

§9.3. Absolute convergence

There are new difficulties when we have to consider series of negative as well as positive terms. For example, although the harmonic series $\sum 1/n$ diverges, if we alternately change signs, the series now converges.

Example 9.18 The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

To see this, we look at the sequences of even partial sums and odd partial sums separately. Since

$$s_{2(n+1)} = s_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2} > s_{2n} \tag{9.51}$$

the sequence of even partial sums is increasing. Similarly,

$$s_{2(n+1)+1} = s_{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} < s_{2n+1} \tag{9.52}$$

tells us that the sequence of odd partial sums is decreasing. Now

$$s_{2n+1} = s_{2n} + \frac{1}{2n+1} > s_{2n}\ , \tag{9.53}$$

that is, the odd partial sums are all greater than all the even partial sums. So both sequences are bounded, and thus converge. But, they converge to the same limit, as we see by taking the limit in equation 9.53:

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} \frac{1}{2n+1} = \lim_{n\to\infty} s_{2n}\ , \tag{9.54}$$

since $1/(2n+1 \to 0)$. Since they both converge to the same limit, the full sequence also converges, and to the same limit.

This argument actually generalizes to any *alternating series*, a series whose terms alternate in sign.

Proposition 9.14 *If a_n is a decreasing sequence, and $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.*

Definition 9.4 *Given a sequence a_n, we say the series $\sum a_n$ converges absolutely if, for the series formed of the absolute values $|a_n|$, we have convergence: $\sum |a_n| < \infty$.*

Proposition 9.15 *If a series converges absolutely, it converges. That is,*

$$\text{If } \sum |a_n| < \infty\ , \quad \text{then } \sum a_n \text{ converges.} \tag{9.55}$$

To see that, let s_n be the nth partial sum of the sequence, p_n the sum of all the positive terms making up s_n, and q_n the sum of the absolute values of all the negative terms. Then

$$s_n = p_n - q_n \; . \tag{9.56}$$

Both sequences p_n and q_n are increasing, and bounded by $\sum |a_n|$, so converge, to, say p, q respectively. Then

$$\sum a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} p_n - \lim_{n\to\infty} q_n = p - q \; . \tag{9.57}$$

Because of this peculiarity of sequences of terms with alternating signs, we shall be most interested in absolute convergence. We can use the tests of section 9.3 (applied to the series of absolute values), to test for absolute convergence.

Example 9.19 $\sum_{n=1}^{\infty} x^n$ converges for $-1 < x < 1$.

This is because the sum of the absolute values is just the geometric series.

Example 9.20 $\sum_{n=1}^{\infty} n^2 x^n$ converges for $-1 < x < 1$.

Here we use the ratio test for the absolute values;

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2 |x|^{n+1}}{n^2 |x|^n} = \left(\frac{n+1}{n}\right)^2 |x| \to |x| \; . \tag{9.58}$$

Thus, we get convergence for x of absolute value less than 1.

§9.4. Power Series

Definition 9.5 *A power series is a series of the form* $\sum_{n=0}^{\infty} a_n (x-c)^n$ *. The point* c *is called the center of the power series.*

A power series defines a function on the set of points for which it converges by

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \; . \tag{9.59}$$

The series provides an effective way of approximately evaluating the function f; our goal in these last sections is to show that most functions do have a power series representation. We can use the ratio test to determine the question of convergence. We take the ratio of successive terms of (4):

$$\frac{|a_{n+1}||x-c|^{n+1}}{|a_n||x-c|^n} = \frac{|a_{n+1}|}{|a_n|} |x-c| \to L|x-c| \; , \tag{9.60}$$

if the limit $L = \lim_{n\to\infty} |a_{n+1}|/|a_n|$ exists. In this case the series converges absolutely for $|x-c| < 1/L$, and diverges for $|x-c| > 1/L$. Thus, the domains of convergence and divergence of the series are separated by the circle, centered at c of radius $1/L$. It can be shown that, in general, there is a circle separating these domains, even if the limit of the ratio of successive coefficients doesn't exist.

Proposition 9.16 *Given the power series representation* $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ *, there is a number* R*,* $0 \le R \le \infty$ *such that we get absolute convergence for all* x*,* $|x-c| < R$*, and divergence for all* x*,* $|x-c| > R$*. R is called the radius of convergence of the power series. We have this value of R:* $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R}$ *, if the limit exists.*

The first example of a power series representation is that of the geometric series:

Example 9.21 $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$. has the radius of convergence $R = 1$.

Example 9.22 $\sum_{n=0}^{\infty} n^k x^n$ converges for $|x| < 1$. for any number k. We use the ratio test. The ratio of successive coefficients is

$$\frac{(n+1)^k}{n^k} = (\frac{n+1}{n})^k \to 1 \tag{9.61}$$

as $n \to \infty$.

Example 9.23 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $\mathrm{R} = \infty$. Using the ratio test:

$$\frac{1}{(n+1)!} \Big/ \frac{1}{n!} = \frac{1}{n+1} \to 0 \ , \tag{9.62}$$

so $R = \infty$, and the series converges for all x. On the other hand, the ratio test shows us that the series

$$\sum_{n=0}^{\infty} n! x^n \tag{9.63}$$

has radius of convergence $R = 0$, so converges only for $x = 0$.

Newton thought of power series as "generalized polynomials" - that is, as polynomials, only longer. This is justified, because we can operate with power series just as we operate with polynomials: we can add, multiply, and substitute in them by doing so term by term.

Example 9.24 $\frac{x}{1-x} = \sum_{n=0}^{\infty} x^{n+1}$ for $\mathrm{R} < 1$. For

$$\frac{x}{1-x} = (x)\frac{1}{1-x} = x(1+x+x^2+x^3+\cdots) = x+x^2+x^3+x^4+\cdots \tag{9.64}$$

Example 9.25 $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$, $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $\mathrm{x} < 1$.

To see the first, we note that $1/(1-x^2)$ is obtained from $1/(1-x)$ by substituting x^2 for x. Thus, the power series representation is obtained in the same way. In the second, we have substituted $-x^2$ for x.

Example 9.26 Find a power series expansion for $1/(5-2x)$ centered at the origin. What is its radius of convergence?

To solve a problem like this, we have to relate the function to another function, whose power series we know. In this case that would be $1/(1-x)$. Now $5-2x = 5(1-(2/5)x)$, so our function is obtained from $1/(1-x)$ by first replacing x by $(2/5)x$, and then dividing by 5. We follow the same instructions with the power series.

Start with

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \; . \tag{9.65}$$

Replace x by $(2/5)x$:

$$\frac{1}{1-(2/5)x} = \sum_{n=0}^{\infty} (\frac{2}{5}x)^n. \tag{9.66}$$

Divide by 5 and clean up:

$$\frac{1}{5-2x} = \frac{1}{5}\sum_{n=0}^{\infty} (\frac{2}{5}x)^n = \sum_{n=0}^{\infty} \frac{2^n x^n}{5^{n+1}} \tag{9.67}$$

We can calculate the radius of convergence using proposition 9.16, or we can reason as follows; since the series we started with converges for $|x| < 1$, our final series converges for $|(2/5)x| < 1$, or $|x| < 5/2$.

Finally, we can also integrate and differentiate power series term by term:

Proposition 9.17 *Suppose that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence R. Then*

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \; , \tag{9.68}$$

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \; , \tag{9.69}$$

and both have the same radius of convergence, R.

Example 9.27 $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$.

We know that the derivative of the arc tangent is $1/(1+x^2)$. Now, in example 9.25, we have already found the power series representation of that function, so we obtain the power series representation of $\arctan x$ by integrating term by term.

Example 9.28 $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x . Let $f(x) = \sum_{n=0}^{\infty} x^n/n!$ Then, differentiating term by term, we find

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \; , \tag{9.70}$$

where the last equation is obtained by replacing the index n by $n+1$. Thus $f'(x) = f(x)$, so satisfies the differential equation, $y' = y$, defining the exponential function. Since $f(0) = 1$ also, it is the exponential function.

Example 9.29 $e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$ for all x. Just replace x in example 9.28 by $-x^2$.

§9.5. Taylor Series

Finally we tackle the question: how do we find the power series representation of a given function? Recalling that the purpose of the power series is to have an effective way to approximate the values of a function by polynomials, we turn to that question: what is the best way to so approximate a function? We start with a function f that has derivatives of all orders defined in an interval about the origin. To begin with, we recall the definition of the derivative in this context:

(9.71)
$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = f'(0) .$$

If we rewrite this as

(9.72)
$$\lim_{x\to 0} \frac{f(x) - (f(0) + f'(0)x)}{x} = 0 ,$$

we see that the linear function $y = f(0) + f'(0)x$ approximates $f(x)$ to first order: $f(0) + f'(0)x$ is closer to $f(x)$ than x is to zero, and by an order of magnitude. We now ask, can we find a quadratic polynomial which approximates f to second order? Let $y = a + bx + cx^2$ be such a polynomial. Then we want

(9.73)
$$\lim_{x\to 0} \frac{f(x) - (a + bx + cx^2)}{x^2} = 0 .$$

We calculate this limit using l'Hôpital's rule. First of all, for l'Hôpital's rule to apply, we have to have $a = f(0)$. Then

(9.74)
$$\lim_{x\to 0} \frac{f(x) - (f(0) + bx + cx^2)}{x^2} =^{l'H} \lim_{x\to 0} \frac{f'(x) - (b + 2cx)}{2x} .$$

We can apply l'Hôpital's rule again, if we have $b = f'(0)$:

(9.75)
$$\lim_{x\to 0} \frac{f(x) - (f'(0) + 2cx)}{2x} =^{l'H} \lim_{x\to 0} \frac{f''(x) - 2c}{2} = 0$$

if $c = f''(0)/2$. We conclude that the polynomial

(9.76)
$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

approximates f to second order: this is closer to $f(x)$ than x is to 0 by two orders of magnitude. Furthermore, it is the unique quadratic polynomial to do so.

We can repeat this procedure as many times as we care to, concluding

Proposition 9.18 *The polynomial which approximates f near 0 to nth order is*

$$f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\cdots+\frac{f^{(n)}(0)}{n!}\ . \tag{9.77}$$

Of course we can make the same argument at any point, not just the origin. To summarize:

Definition 9.6 *Suppose that f is a function with derivatives at all orders defined in an interval about the point c. The Taylor polynomial of degree n of f, centered at c is*

$$(T_c^{(n)}f)(x)=\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k\ . \tag{9.78}$$

Proposition 9.19 *The Taylor polynomial $T_c^{(n)}f$ is the polynomial of degree n which approximates f near c to nth order.*

So, we can compute effective approximations to the values of $f(x)$ near c by these Taylor polynomials; but the question is, how effective is this? More precisely, what is the error? We use this estimate:

Proposition 9.20 *Suppose that f is differentiable to order $n+1$ in the interval $[c-a,c+a]$ centered at the point c. Then the error in approximating f in this interval by its Taylor polynomial of degree n, $T_c^{(n)}f$ is bounded by*

$$\frac{M_{n+1}}{(n+1)!}|x-c|^{n+1}\ , \tag{9.79}$$

where M_{n+1} is a bound of the values of $f^{(n+1)}$ over the interval $[c-a,c+a]$. To be precise, we have the inequality

$$|f(x)-T_c^n f(x)|\le\frac{M_{n+1}}{(n+1)!}|x-c|^{n+1}\ . \tag{9.80}$$

In the next chapter we will show how the error estimate is obtained, and see how to work with it. What we want now is to concentrate on the representation by series.

Definition 9.7 *Let f be a function which is differentiable to all orders in a neighborhood of the point c. The Taylor series for f centered at c is*

$$T_c f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \tag{9.81}$$

If c is the origin, this series is called the Maclaurin series for f.

Proposition 9.21 *Suppose that f is a function which has derivatives of all orders in the interval $(c-a,c+a)$, Let M_n be a bound for the nth derivative of f in the interval. If the sequence*

$$\frac{M_n}{n!}|x-c|^n\to 0, \tag{9.82}$$

converges to zero for all x in the interval, then f is given by its Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \tag{9.83}$$

in $(c-1, c+a)$.

As an example, e^x has the Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} . \tag{9.84}$$

We have already shown by other means. We can verify this using proposition 9.21 as well, since the nth derivative of e^x is still e^x, and the value at $x = 0$ is 1. By a parallel calculation we obtain the power series representation of e^x centered at any point:

Example 9.30 For c any point, the function e^x has the Taylor series representation centered at c:

$$e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n . \tag{9.85}$$

We do have to verify that the remainders converge to zero; that is the terms 9.82 converge to zero. Since e^x is an increasing function, its maximum in the interval $[a-c, a+c]$ is at $x = a+c$, so we can take $M_n = e^{a+c}$. Then, for the exponential function we have

$$\lim_{n\to\infty} \frac{M_n}{n!}|x-c|^n = e^{a+c} \lim_{n\to\infty} \frac{|x-c|^n}{n!} = 0 \tag{9.86}$$

by example 9.24.

It is useful to make the following observation

Proposition 9.22 *Suppose that f has a power series representation:*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n . \tag{9.87}$$

Then, this is its Taylor series. More precisely:

$$a_n = \frac{f^{(n)}(c)}{n!} . \tag{9.88}$$

This is easy to see; if we differentiate 9.87 k times we obtain:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k)a_n(x-c)^{n-k} . \tag{9.89}$$

Now, let $x = c$ and obtain $f^{(k)}(c) = k!a_k$, for all terms but the first have the factor $x - c$.

So, if we have found a power series representative of a function, then that is automatically the Taylors series for the function.

Example 9.31 Find the Maclaurin series for the function $f(x) = 1 - x + 5x^2 - x^3$. Since a polynomial is already expressed as a sum of powers of x, that expression is a power series, and thus the Maclaurin series for the polynomial.

Example 9.32 Find the Taylor series centered at $c = 1$ for the function $f(x) = 1 - x + 5x^2 - x^3$. We have to find the values of the derivatives of f at $c = 1$:

$$f(1) = 4\,, \tag{9.90}$$

$$f'(x) = -1 + 10x - 3x^2\,, \quad \text{so} \quad \mathrm{f}'(1) = 6\,, \tag{9.91}$$

$$f''(x) = 10 - 6x\,, \quad \text{so} \quad \mathrm{f}'(1) = 4\,, \tag{9.92}$$

$$f'''(x) = -6\,, \quad \text{so} \quad \mathrm{f}'(1) = -6\,, \tag{9.93}$$

and all higher derivatives are zero. Thus the Taylor series is

$$f(x) = 4 + 6(x-1) + \frac{4}{2!}(x-1)^2 - \frac{6}{3!}(x-1)^3 = 4 + 6(x-1) + 2(x-1)^2 - (x-1)^3\,. \tag{9.94}$$

Now, we can find the Maclaurin series for many functions, so long as we know how to differentiate them. Following is a list of the most important Maclaurin series.

Proposition 9.23

a) $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n, \quad |x| < 1$

b) $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

c) $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

d) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

e) $\arctan x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

We have already seen how to get (a), (b) and (d). For the trigonometric functions, we proceed as follows. First, the cosine:

$$f(0) = 1, \tag{9.95}$$

$$f'(x) = -\sin x\,, \quad \text{so} \quad \mathrm{f}'(1) = 0\,, \tag{9.96}$$

$$f''(x) = -\cos x\,, \quad \text{so} \quad \mathrm{f}'(1) = -1\,, \tag{9.97}$$

(9.98) $$f'''(x) = \sin x\,, \quad \text{so} \quad \mathrm{f}'(1) = 0\,.$$

(9.99) $$f^{(iv)}(x) = \cos x\,, \quad \text{so} \quad \mathrm{f}^{(\mathrm{iv})}(1) = 1\,.$$

Thus, up to four terms we have

(9.100) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\,.$$

But, now, since we have returned to $\cos x$, the cycle $\{1, 0, -1, 0\}$ repeats itself again and again. We conclude that

(9.101) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots .,$$

which can be rewritten as (iii) of proposition 9.23 above.

One final Taylor series is worth noting: since the integral of $1/(1-x)$ is $-\ln(1-x)$, we can find the Taylor series centered at 1 for $\ln x$ as follows:

(9.102) $$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n\,,$$

(9.103) $$-\ln(1-t) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}\,.$$

Now, make the substitution $x = 1-t$, so $t = 1-x$:

(9.104) $$-\ln x = \sum_{n=0}^{\infty} \frac{(1-x)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}\,.$$