CHAPTER 8

Indeterminate Forms and Improper Integrals

§8.1. L'Hôpital's Rule

To begin this section, we return to the material of section 2.1, where limits are defined. Suppose $f(x)$ is a function defined in an interval around a, but not necessarily at a. Then we write

$$\lim_{x\to a} f(x) = L \tag{8.1}$$

if we can insure that $f(x)$ is as close as we please to L just by taking x close enough to a. If f is also defined at a, and

$$\lim_{x\to a} f(x) = f(a) \tag{8.2}$$

we say that f is *continuous* at a (we urge the reader to review section 2.1). If the expression for $f(x)$ is a polynomial, we found limits by just substituting a for x; this works because polynomials are continuous.

But how do we calculate limits when the expression $f(x)$ cannot be determined at a? For example, we recall the definition of the derivative:

$$f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \; . \tag{8.3}$$

The value cannot be determined by simply evaluating at $x = a$, because both numerator and denominator are 0 at a. This is an example of an *indeterminate form of type* **0/0**: an expression $f(x)/g(x)$, where both $f(a)$ and $g(a)$ are zero. As for 8.3, in case $f(x)$ is a polynomial, we found the limit by long division, and then evaluating the quotient at a (see Theorem 1.1). For trigonometric functions, we devised a geometric argument to calculate the limit (see Proposition 2.7). And as for the rest, we find derivatives using the rules of differentiation.

For the general expression $f(x)/g(x)$ we have

Proposition 8.1 (l'Hôpital's Rule) *If f and g are differentiable at a, and $f(a) = 0$ and $g(a) = 0$, but $g'(a) \neq 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \; . \tag{8.4}$$

This is because

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x-f(a))}{g(x)-g(a)}=\lim_{x\to a}\frac{f(x-f(a)/(x-a))}{g(x)-g(a)/(x-a)}=\frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}=\frac{f'(a)}{g'(a)}\,. \tag{8.5}$$

Each of these equalities can be justified using the hypotheses. It is important when using l'Hôpital's rule to make sure the hypotheses hold; otherwise (see example 8.4 below), we can get the wrong answer.

Example 8.1 $\lim_{x\to 0}\frac{\sin x}{x}=.$

Here the functions are differentiable and both zero at $x=0$, so l'Hôpital's rule applies:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=\cos(0)=1. \tag{8.6}$$

Of course this example is a fake, since we needed to validate this limit just to show the differentiability of $\sin x$.

Example 8.2 $\lim_{x\to 0}\frac{\sin(3x)}{4x}=.$

Both numerator and denominator are 0 at $x=0$, so we can apply l'H (a convenient abbreviation for l'Hôpital's rule):

$$\lim_{x\to 0}\frac{\sin(3x)}{4x}=^{l'H}\lim_{x\to 0}\frac{3\cos(3x)}{4}=\frac{3}{4}\,. \tag{8.7}$$

Example 8.3 $\lim_{x\to 5}\frac{x^2-4x+5}{x-5}=.$

Here both numerator and denominator are zero, so l'H applies:

$$\lim_{x\to 5}\frac{x^2-4x+5}{x-5}=^{l'H}\lim_{x\to 5}\frac{2x-4}{1}=6\,. \tag{8.8}$$

Note that we could also have divided the numerator by the denominator, getting

$$\frac{x^2-4x+5}{x-5}=x+1 \tag{8.9}$$

whose value at $x=5$ is 6.

Example 8.4 $\lim_{x\to 0}\frac{x+2}{3x+1}=.$

Since neither the numerator nor denominator is zero at $x=0$, we can just substitute 0 for x, obtaining 2 as the limit. Note that if we blindly apply l'Hôpital's rule, we get the wrong answer, 1/3.

Example 8.5 $\lim_{x\to 2}\frac{x^3-3x+2}{\tan(\pi x)}=.$

After checking that the hypotheses are satisfied, we get

$$\lim_{x\to 2}\frac{x^3-3x+2}{\tan(\pi x)}=^{l'H}\lim_{x\to 2}\frac{3x^2-3}{\pi\sec^2(\pi x)}=\frac{12-9}{\pi}=\frac{3}{\pi}\,. \tag{8.10}$$

The second limit can be evaluated since both functions are continuous and the denominator nonzero.

Example 8.6 $\lim_{x\to 0} \frac{\sin^2(2x)}{\cos x - 1} = .$

Both numerator and denominator are zero at $x = 0$, so l'Hôpital's rule applies:

$$\lim_{x\to 0} \frac{\sin^2(2x)}{\cos x - 1} =^{l'H} \lim_{x\to 0} \frac{4\sin(2x)\cos(2x)}{-\sin x} . \tag{8.11}$$

Now, numerator and denominator are still zero at $x = 0$, so we can apply l'Hôpital's rule again:

$$=^{l'H} \lim_{x\to 0} \frac{8\cos^2(2x) - 8\sin^2(2x)}{-\cos x} = -8 , \tag{8.12}$$

for now we can take the limit by evaluating the functions.

l'Hôpital's rule also works when taking the limit as x goes to infinity.

Proposition 8.2 *If f and g are differentiable functions, and* $\lim_{x\to\infty} f(x) = 0$ *and* $\lim_{x\to\infty} g(x) = 0$*, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} . \tag{8.13}$$

We see that this is true by the substitution $t = 1/x$, which leads us back to proposition 8.1:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f(1/t)}{g(1/t)} =^{l'H} \lim_{t\to 0} \frac{\frac{-1}{t^2} f'(1/t)}{\frac{-1}{t^2} g'(1/t)} , \tag{8.14}$$

by l'Hôpital's rule and the chain rule. But the factors introduced cancel, so, changing back to $x = 1/t$, we get the proposition.

l'Hôpital's rule works if the limits are infinite (this is called an *indeterminate form of type* ∞/∞):

Proposition 8.3 *If f and g are differentiable functions, and* $\lim_{x\to a} f(x) = \infty$ *and* $\lim_{x\to a} g(x) = \infty$*, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} . \tag{8.15}$$

Here the limit point a may also be infinity.

Example 8.7 $\lim_{x\to \frac{\pi}{2}^-} \frac{\tan x}{\ln(\pi/2 - x)} = .$

The superscript "-" means that the limit is taken from the left; a superscript "+" means the limit is taken from the right. Since both factors tend to ∞, we can use l'Hôpital's rule:

$$\lim_{x\to \frac{\pi}{2}^-} \frac{\tan x}{\ln(\pi/2 - x)} =^{l'H} \lim_{x\to \frac{\pi}{2}^-} \frac{\sec^2 x}{-(\pi/2 - x)^{-1}} = - \lim_{x\to \frac{\pi}{2}^-} \frac{\pi/2 - x}{\cos^2 x} . \tag{8.16}$$

Now, both numerator and denominator tend to 0, so again:

$$=^{l'H} - \lim_{x\to \frac{\pi}{2}^-} \frac{-1}{-2\cos x \sin x} = -\infty , \tag{8.17}$$

since $\cos x \sin x$ is positive and tends to zero.. We leave it to the reader to verify that the limit from the right is $+\infty$.

Example 8.8 $\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x} = .$

This example is here to remind us to simplify expressions, if possible, before proceeding. If we just use l'Hopital's rule directly, we get

(8.18) $$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x} =^{l'H} \lim_{x\to\frac{\pi}{2}^-} \frac{\sec^2 x}{\sec x \tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{\tan x} ,$$

which tells us that the sought-after limit is its own inverse, so is ± 1. We now conclude that since both factors are positive to the left of $\pi/2$, then the answer is +1. But this would have all been easier to use some trigonometry first:

(8.19) $$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}^-} \sin x = 1 .$$

Example 8.9 $\lim_{x\to+\infty} \frac{x^n}{e^x} = .$

Both factors are infinite at the limit, so l'Hopital's rule applies. Let's take the cases $n = 1, 2$ first:

(8.20) $$\lim_{x\to+\infty} \frac{x}{e^x} =^{l'H} \lim_{x\to+\infty} \frac{1}{e^x} = 0 ,$$

(8.21) $$\lim_{x\to+\infty} \frac{x^2}{e^x} =^{l'H} \lim_{x\to+\infty} \frac{2x}{e^x} =^{l'H} 2 \lim_{x\to+\infty} \frac{1}{e^x} = 0 .$$

We see that for a larger integer n, the same argument will work, but with n applications of l'Hôpital's rule. We say that *the exponential function goes to infinity more rapidly than any polynomial*.

Example 8.10 $\lim_{x\to+\infty} \frac{x}{\ln x} = .$

(8.22) $$\lim_{x\to+\infty} \frac{x}{\ln x} =^{l'H} \lim_{x\to+\infty} \frac{1}{1/x} = \lim_{x\to+\infty} x = +\infty .$$

In particular, much as in example 8.9, one can show that polynomials grow more rapidly than any polynomial in $\ln x$.

§8.2. Other Indeterminate Forms

Many limits may be calculated using l'Hôpital's rule. For example: $x \to 0$ and $\ln x \to -\infty$ as $x \to 0$ from the right. Then what does $x \ln x$ do? This is called an *indeterminate form of type* $0 \cdot \infty$, and we calculate it by just inverting one of the factors.

Example 8.11 $\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{1/x} =^{l'H} \lim_{x\to 0} \frac{1/x}{-1/x^2} = -\lim_{x\to 0} \frac{x^2}{x} = -\lim_{x\to 0} x = 0$.

Example 8.12 $\lim_{x\to\infty} x(\pi/2 - \arctan x) =$.

This is of type $0\cdot\infty$, so we invert the first factor:

(8.23) $$\lim_{x\to\infty} x(\pi/2-\arctan x) = \lim_{x\to\infty} \frac{\pi/2-\arctan x}{1/x} =^{l'H} \lim_{x\to\infty} \frac{-1/(1+x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{x^2}{1+x^2}$$

(8.24) $$= \lim_{x\to\infty} \frac{1}{1+x^{-2}} = 1 .$$

Another case, the *indeterminate form* $\infty - \infty$, is to calculate $\lim_{x\to a}(f(x)-g(x))$, where both f and g approach infinity as x approaches a. Although both terms become infinite, the difference could stay bounded, tend to zero, or also tend to infinity. In these cases we have to manipulate the form algebraically to bring it to one of the above forms.

Example 8.13 $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) =$.

(8.25) $$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x} =^{l'H} \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} =^{l'H} \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = 0 .$$

Example 8.14 $\lim_{x\to\infty} x - \sqrt{x^2+20} =$.

Here we can change the subtraction of two positive functions to that of addition by remembering

(8.26) $$x-\sqrt{x^2+20} = (x-\sqrt{x^2+20})\frac{x+\sqrt{x^2+20}}{x+\sqrt{x^2+20}} = \frac{x^2-(x^2+20)}{x+\sqrt{x^2+20}} = \frac{-20}{x+\sqrt{x^2+20}} ,$$

(8.27) $$\lim_{x\to\infty} x-\sqrt{x^2+20} = \lim_{x\to\infty}\frac{-20}{x+\sqrt{x^2+20}} = 0 .$$

Finally, whenever the difficulty of taking a limit is in the exponent, try taking logarithms.

Example 8.15 $\lim_{x\to\infty} x^{1/x} =$.

Let's take logarithms:

(8.28) $$\lim_{x\to\infty}\ln(x^{1/x}) = \lim_{x\to\infty}\frac{1}{x}\ln x = \lim_{x\to\infty}\frac{\ln x}{x} =^{l'H} \lim_{x\to\infty}\frac{1/x}{1} = 0 .$$

Now, exponentiate, using the continuity of exp:

(8.29) $$\lim_{x\to\infty} x^{1/x} = \exp(\lim_{x\to\infty}\ln(x^{1/x})) = e^0 = 1 .$$

§8.3. Improper Integrals: Infinite Intervals

To introduce this section, let us calculate the area bounded by the x-axis, the lines $x=-a$, $x=a$ and the curve $y=(1+x^2)^{-1}$. This is

(8.30) $$\int_{-a}^{a}\frac{dx}{1+x^2}=\arctan x\Big|_{-a}^{a}=2\arctan a\ .$$

Since $\arctan a$ is always less than $\pi/2$, this area is bounded no matter how large we choose a. In fact, since $\lim_{a\to\infty}\arctan a=\pi/2$, the area under the total curve $y=(1+x^2)^{-1}$ adds up to $2(\pi/2)=\pi$. We can write this in the form

(8.31) $$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\pi\ ,$$

using the following definitions.

Definition 8.1 *Suppose that $f(x)$ is defined and continuous for all $x\geq c$. We define*

(8.32) $$\int_{c}^{\infty}f(x)dx=\lim_{a\to\infty}\int_{c}^{a}f(x)dx$$

if the limit on the right exists. In this case we say the integral converges. If there is no limit on the right, we say the integral diverges.

In the same way, if $f(x)$ is defined and continuous in an interval $x\leq c$, we define

(8.33) $$\int_{-\infty}^{c}f(x)dx=\lim_{a\to-\infty}\int_{a}^{c}f(x)dx$$

if the limit exists.

Definition 8.2 *Suppose that $f(x)$ is defined and continuous for all x. Then*

(8.34) $$\int_{-\infty}^{\infty}f(x)dx=\int_{-\infty}^{0}f(x)dx+\int_{0}^{\infty}f(x)dx\ ,$$

if both integrals on the right hand side exist according to definition 8.1.

Note that it is insufficient to define 8.34 by the limit $\lim_{a\to\infty}\int_{-a}^{a}f(x)dx$, for this integral is always zero for an odd function, say $f(x)=x$, and it would not be appropriate to say that such an integral converges.

Example 8.16 $\int_{0}^{\infty}e^{-x}dx=1$.

First we calculate the integral up to the positive number a:

(8.35) $$\int_{0}^{a}e^{-x}dx=-e^{-x}\Big|_{0}^{a}=1-\frac{1}{e^{a}}\ .$$

Now, since $e^{-a}\to 0$ as $a\to\infty$, the limit exists and is 1.

Example 8.17 $\int_{1}^{\infty}x^{-p}dx$ converges for $p>1$.

We calculate the integral over a finite interval:

$$\int_1^a x^{-p}dx = \frac{1}{-p+1}x^{-p+1}\Big|_1^a = \frac{1}{-p+1}(a^{-p+1}-1)\ . \tag{8.36}$$

Now, if $-p+1<0$, $a^{-p+1}\to 0$ as $a\to\infty$, so our conclusion is valid, and in fact

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1} \quad \text{for} \quad p>1\ . \tag{8.37}$$

Also, if $p<1$ then $-p+1>0$, so a^{-p+1} becomes infinite with a, and thus

$$\int_1^\infty \frac{dx}{x^p} \quad \text{diverges for} \quad p<1\ . \tag{8.38}$$

The case $p=1$ cannot be handled this way, because then $-p+1=0$. But

Example 8.18 $\int_1^\infty \frac{dx}{x}$ diverges.

We calculate over a finite interval:

$$\int_1^a \frac{dx}{x} = \ln x\Big|_1^a = \ln a\ , \tag{8.39}$$

which goes to infinity as $a\to\infty$.

Sometimes we can conclude that the improper integral converges, even though we cannot calculate the actual limit. This is because of the following fact:

Theorem 8.1 *Suppose that F is an increasing continuous function of x for all $x\geq c$, and suppose that F is bounded; that is, there is a positive number M such that $M\geq F(x)$ for all x. Then $\lim_{x\to\infty}F(x)$ exists.*

This is an important theorem, known as the *Monotone Convergence Theorem* which is difficult to prove rigorously. To see why it is reasonable at least, consider the *least* upper bound M_0 of the set of values $F(x)$. There must be values $F(x)$ which come as close as we please to M_0, for if not, the values of F stay away from M_0, so this could not be the least upper bound. But now, because F is increasing, that means that eventually all values come that close to M_0.

Example 8.19 $\int_1^\infty e^{-x^2}dx$ converges..

In this range, $x^2\geq x$, so $e^{-x^2}\leq e^{-x}$. So, for any a,

$$\int_1^a e^{-x^2}dx \leq \int_1^a e^{-x}dx \leq 1 \tag{8.40}$$

by example 8.16. Thus the values of the integral are bounded by 1. But since the function is always positive, the integrals increase as a increases. Thus by Theorem 8.1, the limit exists.

This example generalizes to the following

Proposition 8.4 *Suppose that f and g are continuous functions defined for all $x\geq c$, and suppose that for all x, $0\leq f(x)\leq g(x)$. Then*

a) If $\int_c^\infty g(x)dx$ converges, then $\int_c^\infty f(x)dx$ converges.

b) If $\int_c^\infty f(x)dx$ *diverges, then* $\int_c^\infty g(x)dx$ *diverges.*

Example 8.20 $\int_1^\infty \frac{|\cos x|dx}{x^{3/2}}$ converges..

Now, we don't know how to integrate this function, but we do know that $|\cos x| \leq 1$. Thus the integrand is always less than or equal to $x^{-3/2}$, and so, by example 8.17 and proposition 8.4, we can conclude that our integral converges.

§8.4. Improper Integrals: Finite Asymptotes

Now, it is also possible, for a function which has a vertical asymptote, that the values approach the asymptote so fast that the area enclosed is finite.

Example 8.21 Consider $y = x^{-1/2}$ for x positive. For a slightly larger than 0,

$$\int_a^1 x^{-1/2}dx = 2x^{1/2}\Big|_a^1 = 2(1-\sqrt{a}) \ . \tag{8.41}$$

Now, as $a \to 0^+$, this converges to 2. Thus it makes sense to say that $\int_0^1 x^{-1/2}dx = 2$, as we do with this definition.

Definition 8.3 *Let* $f(x)$ *be defined and continuous for all* x *in an interval* $(c,b]$*. We define*

$$\int_c^b f(x)dx = \lim_{a\to c+} \int_a^b f(x)dx \tag{8.42}$$

if the limit exists. Similarly if $f(x)$ *is defined and continuous for all* x *in an interval* $[b,c)$*, we define*

$$\int_b^c f(x)dx = \lim_{a\to c^-} \int_b^a f(x)dx \ . \tag{8.43}$$

Example 8.22 $\int_0^1 x^{-p}dx$ converges for $p < 1$.

We calculate the integral over an interval $(a,1)$, with $a > 0$:

$$\int_a^1 x^{-p}dx = \frac{1}{-p+1}x^{-p+1}\Big|_a^1 = \frac{1}{-p+1}(1-a^{-p+1}) \ . \tag{8.44}$$

Now, if $-p+1 > 0$, $a^{-p+1} \to 0$ as $a \to 0$, so our conclusion is valid, and in fact

$$\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p} \quad \text{for} \quad p < 1 \ . \tag{8.45}$$

Also, if $p > 1$ then $-p+1 < 0$, so a^{-p+1} becomes infinite as a goes to zero, and thus

$$\int_0^1 \frac{dx}{x^p} \quad \text{diverges for} \quad p > 1 \ . \tag{8.46}$$

As for the case $p = 1$, since

(8.47) $$\int_a^1 \frac{dx}{x} = \ln x\Big|_a^1 = -\ln a \ ,$$

this integral diverges to infinity as $a \to 0$. However:

Example 8.23 $\displaystyle\int_0^1 \ln x dx$ converges .

By example 9 of chapter 7, for a positive and near 0,

(8.48) $$\int_a^1 \ln x dx = (x\ln x - x)\Big|_a^1 = -1 - (a\ln a - a) \ .$$

By example 11, chapter 8, $\lim_{a\to 0+} a\ln a = 0$, so the limit exists and is equal to -1.