CHAPTER 7

Techniques of Integration

§7.1. Substitution

Integration, unlike differentiation, is more of an art-form than a collection of algorithms. Many problems in applied mathematics involve the integration of functions given by complicated formulae, and practitioners consult a *Table of Integrals* in order to complete the integration. There are certain methods of integration which are essential to be able to use the Tables effectively. These are: substitution, integration by parts and partial fractions. In this chapter we will survey these methods as well as some of the ideas which lead to the tables. After the examination on this material, students will be free to use the Tables to integrate.

The idea of substitution was introduced in section 4.1 (recall Proposition 4.4). To integrate a differential $f(x)dx$ which is not in the table, we first seek a function $u = u(x)$ so that the given differential can be rewritten as a differential $g(u)du$ which does appear in the table. Then, if $\int g(u)du = G(u) + C$, we know that $\int f(x)dx = G(u(x)) + C$. Finding and employing the function u often requires some experience and ingenuity as the following examples show.

Example 7.1 $\displaystyle\int x\sqrt{2x+1}dx = ?$

Let $u = 2x+1$, so that $du = 2dx$ and $x = (u-1)/2$. Then

$$\int x\sqrt{2x+1}dx = \int \tfrac{u-1}{2}u^{1/2}\tfrac{du}{2} = \frac{1}{4}\int\left(u^{3/2}-u^{1/2}\right)du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C \tag{7.1}$$

$$= \tfrac{1}{30}u^{3/2}(3u-5)+C = \frac{1}{30}(2x+1)^{3/2}(6x-2)+C$$

$$= \frac{1}{15}(2x+1)^{3/2}(3x-1)+C\,, \tag{7.2}$$

where at the end we have replaced u by $2x+1$.

Example 7.2 $\displaystyle\int \tan x dx = ?$

Since this isn't on our tables, we revert to the definition of the tangent: $\tan x = \sin x / \cos x$. Then, letting $u = \cos x$, $du = -\sin x dx$ we obtain

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln u + C = -\ln \cos x + C = \ln \sec x + C \,. \tag{7.3}$$

Example 7.3 $\int \sec x dx = ?$.

This is tricky, and there are several ways to find the integral. However, if we are guided by the principle of rewriting in terms of sines and cosines, we are led to the following:

$$\sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x} \,. \tag{7.4}$$

Now we can try the substitution $u = \sin x$, $du = \cos x dx$. Then

$$\int \sec x dx = \int \frac{du}{1 - u^2} \,. \tag{7.5}$$

This looks like a dead end, but a little algebra pulls us through. The identity

$$\frac{1}{1 - u^2} = \frac{1}{2} \left(\frac{1}{1+u} + \frac{1}{1-u} \right) \tag{7.6}$$

leads to

$$\int \frac{du}{1 - u^2} dx = \frac{1}{2} \int \left(\frac{1}{1+u} + \frac{1}{1-u} \right) du = \frac{1}{2} \left(\ln(1+u) - \ln(1-u) \right) + C \,. \tag{7.7}$$

Using $u = \sin x$, we finally end up with

$$\int \sec x dx = \frac{1}{2} \left(\ln(1 + \sin x) - \ln(1 - \sin x) + C \right) = \frac{1}{2} \ln \left(\frac{1 + \sin x}{1 - \sin x} \right) + C \,. \tag{7.8}$$

Example 7.4 As a circle rolls along a horizontal line, a point on the circle traverses a curve called the *cycloid*. A *loop* of the cycloid is the trajectory of a point as the circle goes through one full rotation. Let us find the length of one loop of the cycloid traversed by a circle of radius 1.

Let the variable t represent the angle of rotation of the circle, in radians, and start (at $t = 0$) with the point of intersection P of the circle and the line on which it is rolling. After the circle has rotated through t radians, the position of the point is as given as in figure 7.1. The point of contact of the circle with the line is now t units to the right of the original point of contact (assuming no slippage), so

$$x(t) = t - \sin t \,, \quad y(t) = 1 - \cos t \,. \tag{7.9}$$

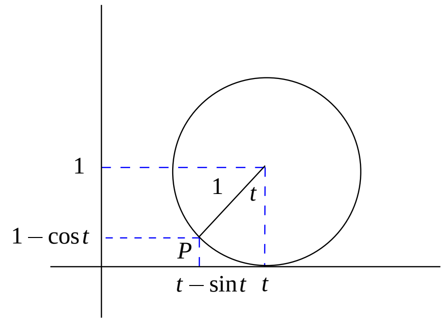
To find arc length, we use $ds^2 = dx^2 + dy^2$, where $dx = (1 - \cos t)dt$, $dy = \sin t dt$. Thus

$$ds^2 = ((1 - \cos t)^2 + \sin^2 t)^2 dt^2 = (2 - 2\cos t)^2 dt^2 \tag{7.10}$$

so $ds = \sqrt{2(1 - \cos t} dt$, and the arc length is given by the integral

$$L = \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos t} dt \,. \tag{7.11}$$

Figure 7.1



To evaluate this integral by substitution, we need a factor of $\sin t$. We can get this by multiplying and dividing by $\sqrt{1+\cos t}$:

$$\sqrt{1-\cos t} = \frac{\sqrt{1-\cos^2 t}}{\sqrt{1+\cos t}} = \frac{|\sin t|}{\sqrt{1+\cos t}}\ . \tag{7.12}$$

By symmetry around the line $t = \pi$, the integral will be twice the integral from 0 to π. In that interval, $\sin t$ is positive, so we can drop the absolute value signs. Now, the substitution $u = \cos t,\ du = -\sin t dt$ will work. When $t = 0,\ u = 1$, and when $t = \pi, u = -1$. Thus

$$L = -2\sqrt{2}\int_1^{-1} u^{-1/2}du = 2\sqrt{2}\int_{-1}^{1} u^{-1/2}du = 2\sqrt{2}(2u^{1/2})\Big|_{-1}^{1} = 8\sqrt{2}\ . \tag{7.13}$$

§7.2. Integration by Parts

Sometimes we can recognize the differential to be integrated as a product of a function which is easily differentiated and a differential which is easily integrated. For example, if the problem is to find

$$\int x\cos x dx \tag{7.14}$$

then we can easily differentiate $f(x) = x$, and integrate $\cos x dx$ separately. When this happens, the integral version of the product rule, called *integration by parts*, may be useful, because it interchanges the roles of the two factors.

Recall the product rule: $d(uv) = udv + vdu$, and rewrite it as

$$udv = d(uv) - vdu \tag{7.15}$$

In the case of 7.14, taking $u = x,\ dv = \cos x dx$, we have $du = dx,\ v = \sin x$. Putting this all in 7.15:

$$x\cos x dx = d(x\sin x) - \sin x dx\ , \tag{7.16}$$

and we can easily integrate the right hand side to obtain

$$\int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x + C \ . \tag{7.17}$$

Proposition 7.1 (Integration by Parts) *For any two differentiable functions u and v:*

$$\int u dv = uv - \int v du \ . \tag{7.18}$$

To integrate by parts:
1. First identify the parts by reading the differential to be integrated as the product of a function u easily differentiated, and a differential dv easily integrated.
2. Write down the expressions for $u,\ dv$ and du, v.
3. Substitute these expressions in 7.18.
4. Integrate the new differential vdu.

Example 7.5 Find $\int xe^x dx$.

Let $u = x,\ dv = e^x dx$. Then $du = dx,\ v = e^x$. 7.18 gives us

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C \ . \tag{7.19}$$

Example 7.6 Find $\int x^2 e^x dx$.

The substitution $u = x^2,\ dv = e^x dx,\ du = 2xdx,\ v = e^x$ doesn't immediately solve the problem, but reduces us to example 3:

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2(xe^x - e^x + C) = x^2 e^x - 2xe^x + 2e^x + C \ . \tag{7.20}$$

Example 7.7 To find $\int \ln x dx$, we let $u = \ln x,\ dv = dx$, so that $du = (1/x)dx,\ v = x$, and

$$\int \ln x dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C \ . \tag{7.21}$$

This same idea works for $\arctan x$: Let

$$u = \arctan x,\ dv = dx \quad du = \frac{dx}{1+x^2},\ v = x \ , \tag{7.22}$$

and thus

$$\int \arctan x = x\arctan x - \int \frac{x}{1+x^2}dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C \ , \tag{7.23}$$

where the last integration is accomplished by the new substitution $u = 1 + x^2,\ du = 2xdx$.

Example 7.8 These ideas lead to some clever strategies. Suppose we have to integrate $e^x \cos x dx$. We see that an integration by parts leads us to integrate $e^x \sin x dx$, which is just as hard. But suppose we integrate by parts again? See what happens:

Letting $u = e^x,\ dv = \cos x dx,\ du = e^x dx, v = \sin x$, we get

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx \ . \tag{7.24}$$

Now integrate by parts again: letting $u = e^x,\ dv = \sin x dx,\ du = e^x dx, v = -\cos x$, we get

$$\int e^x \sin x dx = e^x \cos x + \int e^x \cos x dx \ . \tag{7.25}$$

Inserting this in 7.24 leads to

$$\int e^x \cos x dx = e^x \sin x - e^x \cos x - \int e^x \cos x dx \ . \tag{7.26}$$

Bringing the last term over to the left hand side and dividing by 2 gives us the answer:

$$\int e^x \cos x dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C \ . \tag{7.27}$$

Example 7.9 If a calculation of a definite integral involves integration by parts, it is a good idea to evaluate as soon as integrated terms appear. We illustrate with the calculation of

$$\int_1^4 \ln x dx \tag{7.28}$$

Let $u = \ln x dx,\ dv = dx$ so that $du = dx/x, v = x$, and

$$\int_1^4 \ln x dx = x \ln x\Big|_1^4 - \int_1^4 dx = 4\ln 4 - x\Big|_1^4 = 4\ln 4 - 3 \ . \tag{7.29}$$

Example 7.10

$$\int_0^{1/2} \arcsin x dx = \ \ ?. \tag{7.30}$$

We make the substitution $u = \arcsin x,\ dv = dx,\ du = dx/\sqrt{1-x^2},\ v = x$. Then

$$\int_0^{1/2} \arcsin x dx = x \arcsin x\Big|_0^{1/2} - \int_0^{1/2} \frac{x dx}{\sqrt{1-x^2}} \ . \tag{7.31}$$

Now, to complete the last integral, let $u = 1 - x^2,\ du = -2x dx$, leading us to

$$\int_0^{1/2} \arcsin x dx = \frac{1}{2}\left(\frac{\pi}{6}\right) + \frac{1}{2}\int_1^{3/4} u^{-1/2} du = \frac{\pi}{12} + u^{1/2}\Big|_1^{3/4} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \ . \tag{7.32}$$

§7.3. Partial Fractions

The point of the partial fractions expansion is that integration of a rational function can be reduced to the following formulae, once we have determined the roots of the polynomial in the denominator.

Proposition 7.2
a) $\int \frac{dx}{x-a} = \ln|x-a| + C,$
b) $\int \frac{du}{u^2+b^2} = \frac{1}{b}\arctan\left(\frac{u}{b}\right) + C\ ,$
c) $\int \frac{udu}{u^2+b^2} = \frac{1}{2}\ln(u^2+b^2) + C\ .$

These are easily verified by differentiating the right hand sides (or by using previous techniques).

Example 7.11 Let us illustrate with an example we've already seen. To find the integral

$$\int \frac{dx}{(x-a)(x-b)} \tag{7.33}$$

we check that

$$\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)\ , \tag{7.34}$$

so that

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}\left(\ln|x-a| - \ln|x-b|\right) + C = \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| + C\ . \tag{7.35}$$

The trick 7.34 can be applied to any rational function. Any polynomial can be written as a product of factors of the form $x-r$ or $(x-a)^2+b^2$, where r is a real root and the quadratic terms correspond to the conjugate pairs of complex roots. The partial fraction expansion allows us to write the quotient of polynomials as a sum of terms whose denominators are of these forms, and thus the integration is reduced to Proposition 7.2.

Here is the partial fractions procedure.

1. Given a rational function $R(x)$, if the degree of the numerator is not less than the degree of the denominator, by long division, we can write

$$R(x) = Q(x) + \frac{p(x)}{q(x)} \tag{7.36}$$

   where now $\deg p < \deg q$.
2. Find the roots of $q(x) = 0$. If the roots are all distinct (there are no multiple roots), write p/q as a sum of terms of the form

$$\frac{A}{x-r}\ ,\quad \frac{B}{(x-a)^2+b^2}\ ,\quad \frac{Cx}{(x-a)^2+b^2}\ . \tag{7.37}$$

3. Find the values of $A,\ B,\ C,\ldots.$
4. Integrate term by term using Proposition 7.2.

If the roots are not distinct, the expansion is more complicated; we shall resume this discussion later. For the present let us concentrate on the case of distinct roots, and how to find the coefficients A,B,C in 7.37.

Example 7.12 Integrate $\int \frac{xdx}{(x-1)(x-2)}$.

First we write

(7.38) $$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \ .$$

Now multiply this equation by $(x-1)(x-2)$, getting

(7.39) $$x = A(x-2) + B(x-1) \ .$$

If we substitute $x = 1$, we get $1 = A(1-2)$, so $A = -1$; now letting $x = 2$, we get $2 = B(2-1$, so $B = 2$, and 7.38 becomes

(7.40) $$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2} \ .$$

Integrating, we get

(7.41) $$\int \frac{xdx}{(x-1)(x-2)} = -\ln|x-1| + 2\ln|x-2| + C = \ln\frac{(x-2)^2}{|x-1|} + C \ .$$

So, this is the procedure for finding the coefficients of the partial fractions expansion when the roots are all real and distinct:
1. Write down the expansion with unknown coefficients.
2. Multiply through by the product of all the terms $x - r$.
3. Substitute each root in the above equation; each substitution determines one of the coefficients.

Example 7.13 Integrate $\int \frac{(x^2-3)dx}{(x^2-1)(x-3)}$.

Here the roots are $\pm 1, 3$, so we have the expansion

(7.42) $$\frac{x^2-3}{(x^2-1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-3}$$

leading to

(7.43) $$x^2 - 3 = A(x-1)(x-3) + B(x+1)(x-3) + C(x+1)(x-1) \ .$$

Substitute $x = -1:\ 1-3 = A(-2)(-4)$, so $A = -1/4$.
Substitute $x = 1:\ 1-3 = B(2)(-2)$, so $B = 1/2$.
Substitute $x = 3:\ 9-3 = C(4)(2)$, so $C = 3/4$, and 7.42 becomes

(7.44) $$\frac{x^2-3}{(x^2-1)\,(x-3)} = \left(-\frac{1}{4}\right)\frac{1}{x+1} + \left(\frac{1}{2}\right)\frac{1}{x-1} + \left(\frac{3}{4}\right)\frac{1}{x-3} \ ,$$

and the integral is

(7.45) $$\int \frac{(x^2-3)dx}{(x^2-1)(x-3)} = -\frac{1}{4}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{3}{4}\ln|x-3| + C \ .$$

§7.3.1 Quadratic Factors

Example 7.14 $\int \frac{dx}{x^2-4x-5} = \ ?$

Here we can factor: $x^2-4x-5=(x+1)(x-5)$, so we can write

$$\frac{1}{x^2-4x-5} = \frac{A}{x+1} + \frac{B}{x-5} \tag{7.46}$$

and solve for A and B as above: $A=1/6,\ B=-1/6$, so we have

$$\frac{1}{x^2-4x-5} = \frac{1}{6}\left(\frac{1}{x-5} - \frac{1}{x+1}\right) \tag{7.47}$$

and the integral is

$$\int \frac{dx}{x^2-4x-5} = \frac{1}{6}\ln\left|\frac{x-5}{x+1}\right| + C\ . \tag{7.48}$$

Example 7.15 $\int \frac{dx}{x^2-4x+5} = \ ?$

Here we can't find real factors, because the roots are complex. But we can complete the square: $x^2-4x+5=(x-2)^2+1$, and now use Proposition 7.2b:

$$\int \frac{dx}{x^2-4x+5} = \int \frac{dx}{(x-2)^2+1} = \arctan(x-2)+C\ . \tag{7.49}$$

Example 7.16 $\int \frac{(x+3)dx}{x^2-4x+5} = \ ?.$

Here we have to be a little more resourceful. Again, we complete the square, giving

$$\frac{x+3}{x^2-4x+5} = \frac{x+3}{(x-2)^2+1}\ . \tag{7.50}$$

If only that $x+3$ were $x-2$, we could use Proposition 7.2c, with $u=x-2$. Well, since $x+3=x-2+5$, there is no problem:

$$\begin{aligned}\int \frac{(x+3)dx}{x^2-4x+5} &= \int \frac{(x-2)dx}{(x-2)^2+1} + \int \frac{5dx}{(x-2)^2+1} = \\ &\frac{1}{2}\ln((x-2)^2+1) + 5\arctan(x-2) + C\ .\end{aligned} \tag{7.51}$$

Example 7.17 $\int \frac{(2x+1)dx}{x^2-6x+14} = \ ?.$

First, we complete the square in the denominator: $x^2-6x+14=(x-3)^2+5$. Now, write the numerator in terms of $x-3$: $2x+1=2(x-3)+7$. This gives the expansion:

$$\frac{(2x+1)dx}{x^2-6x+14} = \frac{7}{x^2-6x+14} + 2\frac{x-3}{x^2-6x+14} \tag{7.52}$$

so, using Proposition 7.2:

$$\int \frac{(2x+1)dx}{x^2-6x+14} = 7\int \frac{dx}{(x-3)^2+5} + 2\int \frac{(x-3)dx}{(x-3)^2+5} \tag{7.53}$$

$$= \frac{7}{\sqrt{5}}\arctan\frac{x-3}{\sqrt{5}} + \ln((x-3)^2+5) + C\,. \tag{7.54}$$

Example 7.18 $\int \frac{(x+1)dx}{x(x^2+1)} = \ ?.$

Here we have to expect each of the terms in Proposition 7.2 to appear, so we try an expression of the form

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{B}{x^2+1} + \frac{Cx}{x^2+1}\,. \tag{7.55}$$

Clearing the denominators on the right, we are led to the equation

$$x+1 = A(x^2+1) + Bx + Cx^2\,. \tag{7.56}$$

Setting $x=0$ gives $1=A$. But we have no more roots to substitute to find B and C, so instead we equate coefficients. The coefficient of x^2 on the left is 0, and on the right is $A+C$, so $A+C=0$; since $A=1$, we learn that $C=-1$. Comparing coefficients of x we learn that $1=B$. Thus 7.55 becomes

$$\frac{x+1}{x(x^2+1)} = \frac{1}{x} + \frac{1}{x^2+1} - \frac{x}{x^2+1}\,, \tag{7.57}$$

and our integral is

$$\int \frac{(x+1)dx}{x(x^2+1)} = \ln|x| + \arctan x - \frac{1}{2}\ln(x^2+1) + C\,. \tag{7.58}$$

Example 7.19 $\int \frac{(x^2+1)dx}{x(x^2-4x+5)} = \ ?.$

The denominator is $x((x-2)^2+1)$, so we expect a partial fractions expansion of the form

$$\frac{x^2+1}{x(x^2-4x+5)} = \frac{A}{x} + \frac{B}{(x-2)^2+1} + \frac{C(x-2)}{(x-2)^2+1}\,. \tag{7.59}$$

Clearing of denominators, we obtain the equation

$$x^2+1 = A((x-2)^2+1) + Bx + C(x-2)x\,. \tag{7.60}$$

For $x=0$, we obtain $1=A(5)$, so $A=1/5$. Comparing coefficients of x^2 we obtain $1=A+C$, so $C=-1/5$. Comparing coefficients of x we obtain $0=-4A+B-2C$, so $0=-4/5+B+2/5$, so $B=2/5$ and 7.59 becomes

$$\frac{x^2+1}{x(x^2-4x+5)} = \left(\frac{1}{5}\right)\frac{1}{x} + (\frac{2}{5})\frac{1}{(x-2)^2+1} - \left(\frac{1}{5}\right)\frac{x-2}{(x-2)^2+1}\,, \tag{7.61}$$

which we can integrate to

$$\int \frac{(x^2+1)dx}{x(x^2-4x+5)} = \frac{1}{5}\ln|x| + \frac{2}{5}\arctan(x-2) - \frac{1}{10}\ln(x^2-4x+5) + C\,. \tag{7.62}$$

Multiple Roots

If the denominator has a multiple root, that is there is a factor $x-r$ raised to a power, then we have to allow for the possibility of terms in the partial fraction of the form $1/(x-r)$ raised to the same power. But then the numerator can be (as we have seen above in the case of quadratic factors) a polynomial of degree as much as one less than the power. This is best explained through a few examples.

Example 7.20 $\displaystyle\int \frac{(x^2+1)dx}{x^3(x-1)} = \ ?.$

We have to allow for the possibility of a term of the form $(Ax^2+Bx+C)/x^3$, or, what is the same, an expansion of the form

$$\frac{x^2+1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}\,. \tag{7.63}$$

Clearing of denominators, we obtain

$$x^2+1 = Ax^2(x-1) + Bx(x-1) + C(x-1) + Dx^3\,. \tag{7.64}$$

Substituting $x=0$ we obtain $1 = C(-1)$, so $C=-1$. Substituting $x=1$, we obtain $2=D$. To find A and B we have to compare coefficients of powers of x. Equating coefficients of x^3, we have $0 = A+D$, so $A=-2$. Equating coefficients of x^2, we have $1 = -A+B$, so $B = 1+A = -1$. Thus the expansion 7.63 is

$$\frac{x^2+1}{x^3(x-1)} = -\frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3} + \frac{2}{x-1}\,, \tag{7.65}$$

which we can integrate term by term:

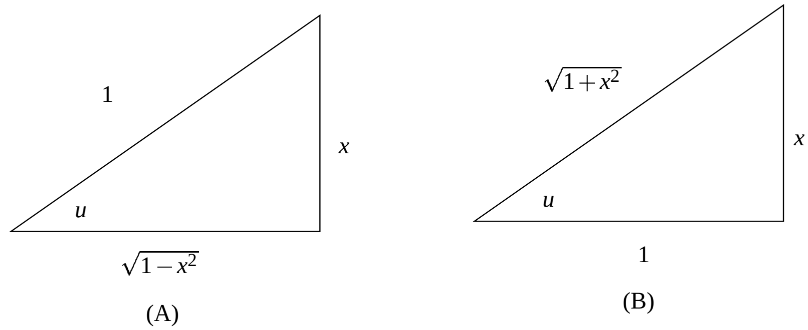
$$\int \frac{(x^2+1)dx}{x^3(x-1)} = -2\ln|x| + \frac{1}{x} + \frac{1}{2x^2} + 2\ln|x-1| + C\,. \tag{7.66}$$

§7.4. Trigonometric Methods

Now, although the above techniques are all that one needs to know in order to use a Table of Integrals, there is one form which appears so often, that it is worthwhile seeing how the integration formulae are found. Expressions involving the square root of a quadratic function occur quite frequently in practice. How do we integrate $\sqrt{1-x^2}$ or $\sqrt{1+x^2}$?

When the expressions involve a square root of a quadratic, we can convert to trigonometric functions using the substitutions suggested by figure 7.2.

Figure 7.2



Example 7.21 To find $\int\sqrt{1-x^2}dx$, we use the substitution of figure 7.2A: $x=\sin u,\ dx=\cos u du,\ \sqrt{1-x^2}=\cos u$. Then

$$\int\sqrt{1-x^2}dx=\int\cos^2 u du\ . \tag{7.67}$$

Now, we use the half-angle formula: $\cos^2 u=(1+\cos 2u)/2$:

$$\int\sqrt{1-x^2}dx=\int\frac{1+\cos 2u}{2}du=\frac{u}{2}+\frac{\sin 2u}{4}+C\ . \tag{7.68}$$

Now, to return to the original variable x, we have to use the double angle formula: $\sin 2u=2\sin u\cos u=x\sqrt{1-x^2}$, and we finally have the answer:

$$\int\sqrt{1-x^2}dx=\frac{\arcsin x}{2}+\frac{x\sqrt{1-x^2}}{4}+C\ . \tag{7.69}$$

Example 7.22 To find $\int\sqrt{1+x^2}dx$, we use the substitution of figure 7.2B: $x=\tan u,\ dx=\sec^2 u du,\ \sqrt{1+x^2}=\sec u$. Then

$$\int\sqrt{1+x^2}dx=\int\sec^3 u du\ . \tag{7.70}$$

This is still a hard integral, but we can discover it by an integration by parts (see Practice Problem set 4, problem 6) to be

$$\int\sec^3 du=\frac{1}{2}(\sec u\tan u+\ln|\sec u+\tan u|)+C\ . \tag{7.71}$$

Now, we return to figure 7.2B to write this in terms of x: $\tan u=x,\ \sec u=\sqrt{1+x^2}$. We finally obtain

$$\int\sqrt{1+x^2}dx=\frac{1}{2}(x\sqrt{1+x^2}+\ln|\sqrt{1+x^2}+x|)+C\ . \tag{7.72}$$

Example 7.23 $\int x\sqrt{1+x^2}dx=\ ?$.

Don't be misled: always try simple substitution first; in this case the substitution $u = 1+x^2$, $du = 2xdx$ leads to the formula

$$\int x\sqrt{1+x^2}dx = \frac{1}{2}\int u^{1/2}du = \frac{2}{3}\left(1+x^2\right)^{3/2} + C\,. \tag{7.73}$$

Example 7.24 $\int x^2\sqrt{1-x^2}dx =$?.

Here simple substitution fails, and we use the substitution of figure 7.2A: $x = \sin u$, $dx = \cos u du$, $\sqrt{1-x^2} = \cos u$. Then

$$\int x^2\sqrt{1-x^2}dx = \int \sin^2 u \cos^2 u du\,. \tag{7.74}$$

This integration now follows from use of double- and half-angle formulae:

$$\begin{aligned}\int \sin^2 u\cos^2 u du = \frac{1}{4}\int \sin^2(2u)\,du = \frac{1}{8}\int (1-\cos(4u))\,du = \\ \frac{1}{8}\left(u - \frac{\sin(4u)}{4}\right) + C\,.\end{aligned} \tag{7.75}$$

Now, $\sin(4u) = 2\sin(2u)\cos(2u) = 4\sin u\cos u(1-2\sin^2 u) = 4x\sqrt{1-x^2}(1-2x^2)$. Finally

$$\int x^2\sqrt{1-x^2}dx = \frac{\arcsin x}{8} + \frac{x\sqrt{1-x^2}(1-2x^2)}{2} + C\,. \tag{7.76}$$

For the remainder of this course, we shall assume that you have a table of integrals available, and know how to use it. There are several handbooks, and every Calculus text has a table of integrals on the inside back cover. There are a few tables on the web:

`http://math2.org/math/integrals/tableof.htm`
`http://www.cahs1.org/lessonIcalc/table`
`/table_of_integrals.htm`
`http://www.engineering.com/community/library/textbook`
`/calculus/calculus_table_integrals_content.htm`
`http://www.maths.abdn.ac.uk/~jrp/ma1002/website/int`
`/node51.html`