CHAPTER 6

Transcendental Functions

§6.1. The Exponential Function

In many dynamical processes we are interested in studying the development of a variable as time progresses where the laws governing the process do not change over time. For example: (1) growth of a bacteria culture, (2) spread of an epidemic, (3) decay of a radioactive material, (4) cooling of a hot metal thrust into water, (5) growth of an interest bearing investment fund. Let $x = x(t)$ be the variable in such a process, where t represents time. In the above examples x would be (1) the mass of the bacteria, (2) the number of infected people, (3) the remaining mass of the original material, (4) the difference between the temperature of the metal and the ambient temperature, (5) the value of the fund. These are examples of *growth processes*. In studying such processes we consider the *per unit rate of growth*:

$$r = \frac{x'(t)}{x(t)}\ , \tag{6.1}$$

which can be written as the differential equation

$$\frac{dx}{dt} = rx\ . \tag{6.2}$$

In many of these processes, the per unit rate of growth r is constant. For our examples: (1) so long as the source of nourishment is plentiful, we can expect the rate at which the bacteria develop to remain unchanged, (3) the physical process of fission depends only on the nature of the atom and is the same throughout the material and remains unchanged over time, (4) the rate at which heat is lost depends only on the temperature difference at the interface and the thermal properties of the material, (5) the growth of an investment account is the interest rate, set periodically by the bank. As for (2), the rate of spread of an epidemic depends upon the nature of the disease, and may depend on time, mortality rate as well as the rate of interaction between infected and non-infected people.

In case r is constant, equation 6.1 leads to a separable differential equation:

$$\frac{dx}{x} = rdt \tag{6.3}$$

which we do not yet know how to integrate. However, since $1/x$ is a continuous function (for $x \neq 0$), there is a solution. We shall return to this integral later; the point here is that the differential equation 6.2 (or 6.3) has a solution.

Definition 6.1 *The exponential function, written $x(t) = e^{rt}$, is the solution of the differential equation 6.2 with value $x(0) = 1$.*

If $r > 0$, r is called the *growth rate*; if $r < 0$, it is the *decay rate*. More precisely, the function e^{rt} is defined by the conditions

$$\frac{d}{dt}e^{rt} = re^{rt} , \qquad e^{r0} = 1 . \tag{6.4}$$

The exponential function, like the trigonometric functions, is a *transcendental function*. These are functions which cannot be expressed as a quotient of polynomials; in this sense they *transcend* rational functions. In due course, we shall find ways to calculate approximate values of the transcendental functions; for us now it suffices to know that these calculations have been done and are incorporated into our calculators.

Proposition 6.2 *The solution of the differential equation $x'(t) = rx(t)$, with initial value $x(0) = x_0$ is*

$$x(t) = x_0 e^{rt} . \tag{6.5}$$

To verify this, we show that the function defined by 6.5 solves this initial value problem. First, $x(0) = x_0 e^{r0} = x_0$. Differentiating:

$$x'(t) = x_0 \frac{d}{dx} e^{rt} = x_0 r e^{rt} = rx(t) . \tag{6.6}$$

Example 6.1 \$500 is deposited in an account, continuously compounded at an interest rate of 5% per year. What is the value of the account after 5 years?

Let $x(t)$ be the value of the fund at time t. Then $x(0) = 500$. The phrase *continuously compounded* tells us that the fund grows continuously at the given rate, so x satisfies the differential equation 6.2 with $r = 0.05$. The solution then is

$$x(t) = 500e^{0.05t} . \tag{6.7}$$

At $t = 5$, we calculate: $e^{.05(5)} = e^{0.25} = 1.284$, so $x(5) = 500(1.284) = 642.01$ dollars.

Example 6.2 According to the census, the US population in 2000 was 281.4 million. The growth rate over the preceding decade was 0.1235. Assuming that growth rate continues during the present century, what will be the US population in 2050 ?

Let $x(t)$ be the population of the US in millions, where t is the number of decades after 2000. Then $x(0) = 281.4$, and Proposition 6.2 tells us

$$x(t) = 281.4e^{(0.1235)t} \tag{6.8}$$

At 2050, $t = 5$, so the answer is $x(5) = 281.4e^{0.1235(5)} = 521.8$ million.

Example 6.3 The radioactive isotope $^{128}\mathrm{I}$ has a decay rate of 0.0279 per minute. How many grams of an initial 100g supply of $^{128}\mathrm{I}$ remain after 20 minutes?

Let $x(t)$ be the amount of ^{128}I present after t minutes. The information we are given is that $x(0) =$ 100 g and $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} = -0.0279x(t)\ . \tag{6.9}$$

Thus, by Proposition 6.2, $x(t) = 100e^{(-0.0279)t}$, and the answer is

$$x(20) = 100e^{-0.0279(20)} = 57.23 \text{ g}. \tag{6.10}$$

We can come upon a way to calculate the exponential function by starting with a comparison of continuously compounded interest with other ways of compounding interest. Suppose I deposit one dollar in a bank account with an interest rate of 10% per year. If the rate is *simple*, that is, it is paid out once only at the end of the year, then at that time the account will have \$1.10. This is considered unfair of the bank, since they have had the use of my dollar throughout the year and have been investing it over and over, but they have transferred my share of the earning only at the end of the year. Suppose instead the bank paid me twice a year, and the amount added to my account after six months were reinvested. Then, after 6 months, I have \$1.05. This is reinvested for another half year, so now, at the end of the year I accrue another $1.05(0.05) = 0.0525$, so I will have \$1.1025.

Now, suppose that the interest rate is r per year, and is paid in n periods per year. Then, in each period I gain r/n of the amount I had at the beginning of the period. Let $P(k)$ represent the amount I have at the end of k periods. I start with $P(0)$ dollars. The law of change here is $P(k) = P(k-1) + (r/n)P(k-1)$: my increment in any period is r/n times the amount at the beginning of the period. Thus

$$P(1) = P(0) + P(0)(r/n) = P(0)\left(1 + \frac{r}{n}\right)\ , \tag{6.11}$$

$$P(2) = P(1)\left(1 + \frac{r}{n}\right) = P(0)\left(1 + \frac{r}{n}\right)^2\ , \tag{6.12}$$

and after k periods,

$$P(k) = P(0)\left(1 + \frac{r}{n}\right)^k\ . \tag{6.13}$$

Now, let t be the time (in years) the fund is allowed to grow at this interest rate. The number of periods is $k = nt$, so we can rewrite 6.13 as

$$P(t) = P(0)\left(1 + \frac{r}{n}\right)^{nt}\ . \tag{6.14}$$

Now, as we let the number of periods get larger and larger, this approaches continuous compounding in the limit; that is, formula 6.14 approaches the formula

$$P(t) = P(0)e^{rt}\ . \tag{6.15}$$

We conclude, taking $P(0) = 1$:

Proposition 6.3

$$e^{rt} = \lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{nt}\ . \tag{6.16}$$

In particular

$$e = e^1 = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n , \tag{6.17}$$

which is approximately $2.71828\ldots$.

Example 6.4 \$500 is invested at 5% per year, compounded quarterly. What is the value of the fund at the end of 5 years?

We use formula (5) with $P(0) = 500,\ r = 0.05,\ n = 4, t = 5$:

$$P(5) = 500\left(1+\frac{0.05}{4}\right)^{20} = 641.01 . \tag{6.18}$$

Notice that this is close to, but less than the answer to example 6.1: 642.01. If the compounding is done only annually, then the answer would be $500(1+0.05)^5 = 638.14$. Since the result using simple interest is just 600, clearly any kind of compounding is preferable, and quarterly compounding is already very close to continuous compounding.

We use exponential notation to denote the exponential function $y = e^x$ because it obeys the rules of exponents:

Proposition 6.4 *For any two numbers A and B:*

$$\text{a)}\quad e^{A+B} = e^A \cdot e^B ,\ \text{b)}\quad e^{AB} = (e^A)^B ,\ \text{c)}\quad e^{-A} = \frac{1}{e^A} \tag{6.19}$$

Let's start with c). Let $y = 1/e^x$. Then, by the chain rule:

$$\frac{dy}{dx} = \frac{-1}{(e^x)^2}\frac{d}{dx}e^x = \frac{-1}{(e^x)^2}e^x = \frac{-1}{e^x} = -y . \tag{6.20}$$

Since $y(0) = 1$, y solves the initial value problem of Proposition 6.2 with $r = -1$, so $y = e^{-x}$; that is, $1/e^x = e^{-x}$.

To show a), let B be a number, and consider $y = e^{x+B}e^{-x}$. By the product rule for differentiation:

$$\frac{dy}{dx} = e^{x+B}(-e^{-x}) + e^{x+B}e^{-x} = 0 , \tag{6.21}$$

so y is constant. But at $x = 0$, $y = e^B$, so

$$e^{x+B}e^{-x} = e^B . \tag{6.22}$$

Now, replacing x by A, and using c), this says $e^{A+B}/e^A = e^B$, which is a). b) is proven the same way: differentiate the function $y = (e^x)^B$:

$$\frac{dy}{dx} = B(e^x)^{B-1}\cdot e^x = B(e^x)^B , \tag{6.23}$$

and $y(0) = 1$, so y is the solution function of Proposition 6.2 with $r = B$.

Figure 6.1

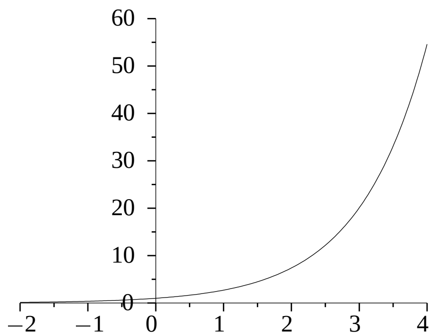
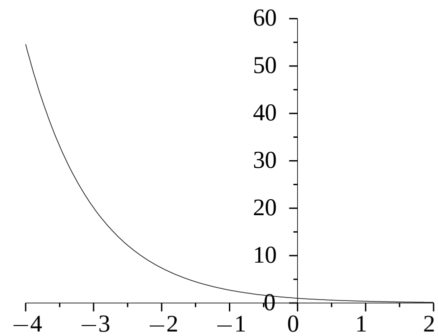


Figure 6.2



In particular, since $e^x = (e^1)^x$, we can think of the exponential function as the number e of 6.17 raised to the xth power.

Proposition 6.5

a) The exponential function $y = e^x$ is a strictly increasing function.

b) $\lim_{x\to\infty} e^x = +\infty$*; in particular $e^n \geq 2^n$ for all n.*

c) $\lim_{x\to-\infty} e^x = 0$.

a) follows from the fact that $e^x > 0$ for all x, so the derivative of e^x is always positive. To see b), we first observe, from the binomial expansion,

$$(1+\frac{1}{n})^n = 1 + n\frac{1}{n} + \cdots > 2\ , \tag{6.24}$$

so in the limit $e \geq 2$. Thus $e^n \geq 2^n$ for all n. Now, for c):

$$\lim_{x\to-\infty} e^x = \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0\ . \tag{6.25}$$

Since the second derivative of $y = e^x$ is again e^x which is positive, the curve is always concave up. From this information, we can easily sketch the graph of the exponential function (see figure 6.1). To graph the function $y = e^{-x}$, we just reflect in the y-axis (see figure 6.2).

Since the derivative of e^x is itself, so is its indefinite integral:

Proposition 6.6 $\int e^x dx = e^x + C\ .$

Example 6.5 Find $\int_0^3 xe^{x^2} dx$.

Let $u = e^{x^2}$, so that (by the chain rule) $du = 2xe^{x^2} dx$. When $x = 0$, $u = 1$ and when $x = 3$, $u = e^9$. Thus

$$\int_0^3 xe^{x^2} dx = \frac{1}{2}\int_1^{e^9} du = \frac{1}{2}u\Big|_1^{e^9} = \frac{1}{2}\left(e^9 - 1\right)\ . \tag{6.26}$$

§6.2. The Logarithm

At what rate (compounded continuously) should I invest \$500 so as to have \$800 in the account at the end of 6 years? If r is the unknown rate, the answer is given by solving the equation:

$$800 = 500e^{6r} . \tag{6.27}$$

To answer this, we have to find the number a such that $e^a = 8/5$. That is, we have to *invert* the operation of exponentiation. This is done by the logarithm.

Definition 6.2 *Given a positive number x, the natural logarithm of x, denoted $y = \ln x$ is that number y such that $e^y = x$.*

Any positive number x lies between 0 and ∞, so by the intermediate value theorem applied to the exponential function and Proposition 6.5, there is a number y such that $e^y = x$. Since the exponential function is strictly increasing, there is only one such number. Thus the above definition makes sense. The logarithm is a transcendental function whose values have been calculated and are stored in our calculators.

Example 6.6 To conclude the above discussion, we solve 6.14: $e^{6r} = 8/5$, so $6r = \ln(8/5) = 0.47$, so $r = 0.47/6 = 0.078$: the interest rate must be 7.8%.

Example 6.7 How long does it take for a quantity of ^{128}I to be reduced to half its size?

Referring to example 6.3, if we start with an amount $P(0)$ of ^{128}I , the amount we have after t minutes is $P(t) = P(0)e^{-(0.0279)t}$ To solve our problem we find t such that $0.5P(0) = P(0)e^{-(0.0279)t}$, or $0.5 = e^{(0.0279)t}$ This leads to $0.0279t = -\ln(1/2)$ or $t = 24.84$ minutes. This time is called the *half-life* of ^{128}I.

Example 6.8 The rate of decay of the radioactive isotope of carbon (^{14}C) is 1.211×10^{-4} per year. In how many years will it take a certain amount of ^{14}C to be reduced by 10%?

Let $C(t)$ represent the amount of ^{14}C in t years. We can take $C(0) = 1$, and the question is: for what T is $C(T) = 0.9$? Since $C(t)$ satisfies the differential equation

$$C'(t) = -1.211 \times 10^{-4} C(t) . \tag{6.28}$$

We conclude (Definition 6.4)

$$C(t) = e^{-1.211 \times 10^{-4} t} \tag{6.29}$$

so we must solve

$$0.9 = e^{-1.211 \times 10^{-4} T} \tag{6.30}$$

Then

$$T = \frac{\ln(0.9)}{-1.211 \times 10^{-4}} = \frac{-0.1054}{-1.211} \times 10^4 = 870 \text{ years} . \tag{6.31}$$

Example 6.9 At the time an organic material is buried, its ^{14}C content ceases to be replenished by cosmic radiation, so is subject only to the radioactive decay, as described in example 6.8. Suppose the carbon content of a fossil is discovered to contain 84% of the amount of ^{14}C had it not been buried. How old is it?

Let time $t = 0$ represent the time the fossil was buried, and T the number of years since then. If we take $P(0) = 1$, then $P(T) = 0.84$, so we must solve the equation

$$0.84 = e^{-1.211\times 10^{-4}T}\ , \tag{6.32}$$

or

$$T = \frac{\ln(0.84)}{-1.211\times 10^{-4}} = \frac{-0.1744}{-1.211}\times 10^4 = 1440 \text{ years}\ . \tag{6.33}$$

Example 6.10 The half life of a radioactive element is the time it takes for any amount to decrease to half its original size. To find the half-life T of ^{14}C, we solve

$$\frac{1}{2} = e^{-1.211\times 10^{-4}T}\ . \tag{6.34}$$

The answer is $T = \ln 2/(1.211\times 10^{-4}) = 5724$ years.

We now look into the properties of the logarithmic function:

Proposition 6.7 $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$.

Let $y = \ln x$, so that $x = e^y$. Taking differentials: $dx = e^y dy$, so

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}\ . \tag{6.35}$$

The properties (Proposition 6.4) of the exponential function translate into properties of the logarithm:

Proposition 6.8 *For any two positive numbers A and B:*
a) $\ln(AB) = \ln A + \ln B$
b) $\ln(A^B) = B(\ln A)$
c) $\ln\frac{1}{A} = -\ln A$.

For c), let $a = \ln A$, so that $A = e^a$. Then, from proposition 6.4: $e^{-a} = 1/e^a = 1/A$, so $\ln(1/A) = -a = -\ln A$. For a), let $b = \ln B$, so that $B = e^b$. Then, again, from proposition 6.4: $AB = e^a e^b = e^{a+b}$, so $a + b = \ln(AB)$, which is a). b) can be shown in the same way.

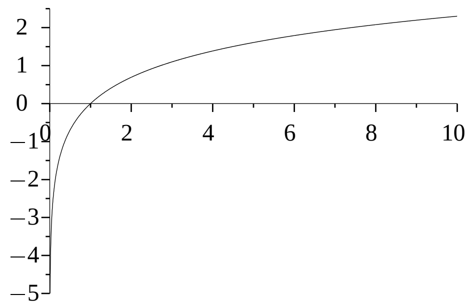
Proposition 6.9
a) $\ln x$ *is a strictly increasing function of* x *for* $x > 0$.
b) $\lim_{x\to\infty}\ln x = \infty$
c) $\lim_{x\to 0}\ln x = -\infty$

These follow directly from the corresponding assertions for the exponential (Prop 6.5). This information suffices to sketch the graph of $y = \ln x$ (figure 6.3).

We note that for $x < 0$, the function $y = \ln|x| = \ln(-x)$ satisfies the differential equation $dy/dx = 1/x$:

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{-x}(-1) = \frac{1}{x}\ . \tag{6.36}$$

Figure 6.3: $y = \ln x$



This then tells us what the indefinite integral of $1/x$ is (for $x \neq 0$):

Proposition 6.10 $\displaystyle\int \frac{dx}{x} = \ln|x| + C \quad \text{for} \quad x \neq 0\ .$

Example 6.11 $\displaystyle\int_2^5 (2x+1)^{-1} dx = \ ?$

Let

$$u = 2x+1,\ du = 2dx. \tag{6.37}$$

At $x = 2,\ u = 5$, and at $x = 5,\ u = 11$. Thus

$$\int_2^5 (2x+1)^{-1}\, dx = \frac{1}{2}\int_5^{11} \frac{du}{u} = \frac{1}{2}\ln u\,\Big|_5^{11} = \frac{1}{2}\ln\frac{11}{5}\ . \tag{6.38}$$

Example 6.12 $\displaystyle\int \frac{e^x}{e^x+1} dx = \ ?$

Let $u = e^x,\ du = e^x dx$. Then

$$\int \frac{e^x}{e^x+1} dx = \int \frac{du}{u} = \ln u = \ln(e^x+1) + C\ . \tag{6.39}$$

Example 6.13 Find the solution of the differential equation $dy/dx = xy$, $y(0) = 1$.

The equation is separable and becomes: $dy/y = xdx$. Integrating both sides gives $\ln y = x^2/2 + C$. Substituting $x = 1,\ y = 0$ we find $\ln 1 = 0 + C$, so $C = 0$. Thus the solution is given by

$$\ln y = x^2/2\ , \quad \text{or} \quad y = e^{x^2/2}\ . \tag{6.40}$$

Inhibited growth

The growth equation $dx/dt = rx$ does not really work for biological populations, since they do not appear to continue to grow exponentially without bound. In fact, there are always factors present which cause the growth rate to decrease as the population increases. One such is the competition over nutrient: the growth rate of bacteria in an agar dish diminishes as the population increases. A good model for this is to let the growth rate r decrease linearly as a function of x: $r = a - bx$, where a is the genetic growth factor, and b is the *inhibiting factor*. In this model, the growth equation is replaced by

$$\frac{dx}{dt} = (a-bx)x\ . \tag{6.41}$$

Notice that if $x = a/b$ we have $dx/dt = 0$, so there is no growth. This is called the *stable* population. Equation 6.41 (called the *logistic equation*) is separable and can be rewritten

$$\frac{dx}{(a-bx)x} = dt\ . \tag{6.42}$$

To solve this we need a little algebra:

$$\frac{1}{(a-bx)x} = \frac{1}{a}\left[\frac{1}{x} + \frac{b}{a-bx}\right] \tag{6.43}$$

so

$$\int \frac{dx}{(a-bx)x} = \frac{1}{a}\int \frac{dx}{x} + \frac{b}{a}\int \frac{dx}{a-bx} = \frac{1}{a}(\ln x - \ln(a-bx)) + C\ . \tag{6.44}$$

Thus 6.42 becomes

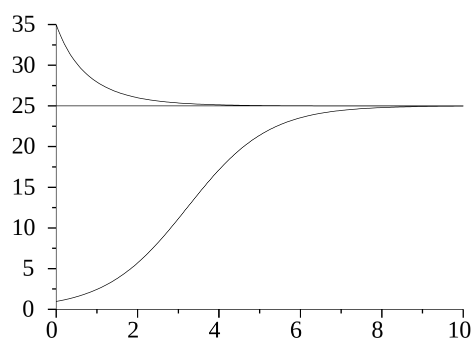
$$t + C = \frac{1}{a}\ln\left(\frac{x}{a-bx}\right) \tag{6.45}$$

or, after exponentiating, $Ke^{at} = x/(a-bx)$. If we solve for x in terms of t, we obtain

$$x = \frac{aKe^{at}}{1+bKe^{at}} = \frac{aK}{bK+e^{-at}}\ . \tag{6.46}$$

Since $dx/dt = x(a-bx)$, this function is increasing in the interval $x < a/b$, and is decreasing for $x > a/b$. Finally, from the second form in 6.46, we see that $x \to a/b$ as $t \to +\infty$. In figure 6.4, we have drawn three typical solutions of the logistic differential equation.

Figure 6.4: The logistic equation.



Example 6.14 A certain bacterium grows at the growth rate of 80% per hour. In a particular agar dish, the inhibiting factor is 0.002. What is the stable population? If 100 g of the bacterium is put in the dish, what will be the size of the population after 2 hours?

Here $a = 0.8$, and $b = 0.002$, and the growth equation is

$$x' = (0.8 - 0.002x)x\,. \tag{6.47}$$

The stable populations is 0.8/0.002 = 400 g. To find the population after 2 hours, we have to solve for K in equation 6.46:

$$x = \frac{0.8K}{0.002K + e^{-0.8t}}\,, \tag{6.48}$$

at the initial value $x = 100$ when $t = 0$. This leads to the equation $100(0.002K + 1) = 0.8K$, which solves to $K = 166.7$. Then

$$x(2) = \frac{0.8(166.7)}{0.002(166.7) + e^{-0.8(2)}}, \tag{6.49}$$

giving $x(2) = 249.25$ g. If there were no inhibition to the growth the population would be $100e^{1.6} = 495$ g.

Example 6.15 In Example 6.2, we saw that the growth rate of the US population over the decade 1990-2000 was 0.1235. However, if we looked at the census over the previous few decades, we would find that the growth rate was decreasing. If we attribute that to an inhibiting factor, we would be able to estimate that factor to be about 0.00032. Given these data, what is the stable population of the US, and what will the population be in 2050 according to this model?

Let $P(t)$ be the US population t decades after 2000. We start with the differential equation $P' = (0.1235 - 0.00032P)P$. The stable population is $P = 0.1235/0.00032 = 385.93$ million. Now, to estimate the actual population in 2050, we have to evaluate the constant in the general solution

$$P(t) = \frac{0.1235K}{0.00032K + e^{-0.1235t}}\,. \tag{6.50}$$

At $t = 0$, $P = 281.4$: this gives $K = 8400$. Then, the population in 2050 is

$$P(5) = \frac{0.1235(8400)}{0.00032(8400) + e^{-0.1235(5)}} = 321.45 \tag{6.51}$$

million; a much more realistic estimate than the 521.8 million estimate of example 6.2.

§6.3. General Exponentials and Logarithms

We can raise any positive number a to any power p: since $a = e^{\ln a}$, $a^p = (e^{\ln a})^p = e^{p(\ln a)}$. This observation allows us to introduce the general exponential and logarithmic functions.

Definition 6.3 *For a any positive number, we define the exponential function with base a by*

$$a^x = e^{(\ln a)x} \tag{6.52}$$

and the logarithmic function with base a as its inverse function:

$$y = \log_a x \quad \text{if and only if} \quad x = a^y \ . \tag{6.53}$$

To find a formula for $\log_a$, we note that if if $x = a^y$, then $x = e^{(\ln a)y}$, so $(\ln a)y = \ln x$. Replacing y by $\log_a x$, we have

$$\log_a x = \frac{\ln x}{\ln a} \ . \tag{6.54}$$

From the chain rule, we obtain these formula for the derivatives and integrals of these new functions:

Proposition 6.11

$$\frac{d}{dx} a^x = (\ln a)a^x \ , \quad \frac{d}{dx} \log_a x = \frac{1}{(\ln a)x} \ , \tag{6.55}$$

$$\int a^x dx = \frac{a^x}{\ln a} \ . \tag{6.56}$$

This notation allows us to replace the rate of decay of a radioactive element by its half life. For suppose that a certain element has a rate of decay r, and a half-life T. Then $1/2 = e^{-rT}$, so that $r = \ln 2/T$. Now, if an amount A of the element decays for t years, then the amount remaining is

$$A(t) = Ae^{-rt} = Ae^{-\ln 2(t/T)} = A(2^{-t/T}) \ . \tag{6.57}$$

Example 6.16 Suppose that the half-life of a certain element is 40 years. How much will remain of a 1 kg sample after 200 years? After 50 years?

In 200 years the sample will have halved 5 times, so what will remain is $1/2^5$ of a kilogram, or 31.25 g. After 50 years, we have $A(t) = 1000(2^{-50/40}) = 435$ grams.

§6.4. First Order Linear Differential Equations

Definition 6.4 *A first order linear differential equation is one of the type*

$$\frac{dy}{dx} + P(x)y = Q(x) \ . \tag{6.58}$$

It is said to be homogeneous if the function $Q(x)$ is 0.

The equation is said to be first order since it involves only the first derivative, and linear since the equation expresses the first derivative of the unknown function y as a linear function of y.

If P and Q are constant functions we can easily solve the differential equation by separation of variables.

Example 6.17 To solve, say

$$\frac{dy}{dx} = 2y - 3 \tag{6.59}$$

we rewrite the equation in the form $(2y-3)^{-1}dy = dx$. These differentials integrate to the relation

$$\frac{1}{2}\ln(2y-3) = x + C \quad \text{or} \quad \sqrt{2y-3} = Ke^x \,. \tag{6.60}$$

Squaring both sides and solving for y, we get the general solution

$$y = \frac{Ke^{2x}+3}{2} \,. \tag{6.61}$$

For example, to find the solution with initial value $y(0) = 5$, we first solve for K:

$$5 = \frac{Ke^{2(0)}+3}{2} \,, \tag{6.62}$$

so $K = 7$, and the particular solution is $y = (7e^{2x}+3)/2$.

The acute reader will object that the integral of $(2y-3)^{-1}dy$ is $(1/2)\ln|2y-3|$, and if we follow through with this, this seems to lead to the alternative solution

$$y = \frac{3-Ke^{2x}}{2} \,. \tag{6.63}$$

However, this is the same as 6.61, just with a different choice for the constant K. If we use 6.63 with the same initial conditions $y(0) = 5$, we find this $K = -7$, giving the same final answer. For this reason it is often the case that the absolute value is ignored.

The homogeneous equation is separable:

Example 6.18 Solve $y' - 2xy = 0,\ y(2) = 1$.

We separate the variables: $y^{-1}dy = 2xdx$ and integrate:

$$\ln y = x^2 + C \,. \tag{6.64}$$

Substituting the initial condition allows us to solve for C: $\ln 1 = 4 + C$, so $C = -4$. Thus the particular solution is given by

$$\ln y = x^2 - 4 \tag{6.65}$$

which exponentiates to

$$y = e^{x^2-4} \,. \tag{6.66}$$

The general equation $y' + P(x)y = Q(x)$ is solved by first solving the corresponding homogeneous equation $y' + P(x)y = 0$, getting

$$y = Ke^{-\int P dx} \,. \tag{6.67}$$

If we now replace the unknown constant K by an unknown function u, and substitute this into the original equation, we end up with this differential equation for u:

(6.68) $$\frac{du}{dx}e^{-\int P\,dx} = Q(x)$$

which is another separable equation, so is solved by integration. The terms involving an undifferentiated u disappear precisely because $e^{-\int P\,dx}$ solves the homogeneous equation. For this reason $e^{-\int P\,dx}$ is called an *integrating factor*. This method is called that of *variation of parameters*; the idea being to first find the general solution of an easier equation, and then trying that in the original equation, but with the constant replaced by a new unknown function. This method is extremely productive in solving very general types of differential equations.

Example 6.19 Solve $y' - 2xy = x,\ y(0) = 2$.

First solve the homogeneous equation $y' - xy = 0$, leading to

(6.69) $$y = Ke^{x^2} .$$

Now substitute $y = ue^{x^2}$ into the original equation to obtain

(6.70) $$u'e^{x^2} = x \quad \text{or} \quad u' = xe^{-x^2} .$$

This integrates to

(6.71) $$u = -\frac{1}{2}e^{-x^2} + C ,$$

so that our general solution is $y = ue^{x^2}$ with this u:

(6.72) $$y = \left(-\frac{1}{2}e^{-x^2} + C\right)e^{x^2} = -\frac{1}{2} + Ce^{x^2} .$$

Putting in the initial conditions, we get

(6.73) $$2 = -\frac{1}{2} + C$$

so that $C = 5/2$. Thus the solution is

(6.74) $$y = \frac{-1 + 5e^{x^2}}{2}.$$

Example 6.20 Find the general solution to $xy' + y = x^2$.

We first must put this in the form 6.58:

(6.75) $$\frac{dy}{dx} + \frac{y}{x} = x .$$

The solution to the homogeneous equation is $y = Kx$. So, we try $y = ux$, and obtain the equation

(6.76) $$u'x = x ,$$

which has the general solution $u = x + C$. Thus the general solution to the original problem is

$$y = ux = (x + C)x = x^2 + Cx \ . \tag{6.77}$$

Remember the steps to solve the equation $y' + P(x)y = Q(x)$:

1. Solve the homogeneous equation $y' + P(x)y = 0$, obtaining $y = e^{-\int P dx}$.
2. Try the solution $y = ue^{-\int P dx}$, leading to the equation for u : $u'e^{-\int P dx} = Q(x)$, or $u' = Q(x)e^{\int P dx}$.
3. Solve for u, and put that solution in the equation $y = ue^{-\int P dx}$. If an initial value is specified, now solve for the unknown constant.

A useful fact to know about linear first order equations is that if we know one particular solution, then we only have to solve the homogeneous equation to find all solutions.

Proposition 6.12 *Suppose that y_p is a solution of the differential equation $y' + Py = Q$. Then every solution is of the form*

$$y = y_p + Ke^{-\int P dx} \ ; \tag{6.78}$$

that is, every solution is of the form $y_p + y_h$, where y_h is a solution of the homogeneous equation.

For suppose that y is any solution of the equation: $y' + Py = Q$. Then $(y - y_p)' + P(y - y_p) = (y' + Py) - (y'_p + Py_p) = Q - Q = 0$, so solves the homogeneous equation.

Example 6.21 Find the solution of the equation $y' - 2y = 5$ such that $y(0) = 1$.

Now the constant function $y_p = 5/2$ solves the equation, since $y'_p = 0$. The general solution of the homogeneous equation is $y = Ke^{2x}$, so our solution is of the form $y = (5/2) + Ke^{2x}$. Substituting $y = 1,\ x = 0$, we find $1 = 5/2 + K$, so $K = -3/2$, and the particular solution we want is

$$y = \frac{1}{2}(5 - 3e^{2x}) \ . \tag{6.79}$$

Example 6.22 A body falling due to gravity through a fluid is subjected to a resistance of the fluid proportional to its velocity. (Here we are assuming that the density of the body is much higher than the density of the fluid, and that its shape is not relevant). Let $x(t)$ represent the distance fallen at time t and $v(t)$ the velocity. The hypothesis leads to the equation

$$\frac{dv}{dt} = -kv + g \tag{6.80}$$

for some constant k (g is the acceleration of gravity), called the coefficient of resistance of the fluid. Notice that the constant $v = g/k$ is a solution of the equation. This is called the “free fall velocity”; any falling body will accelerate until it reaches this maximum velocity. By proposition D.2, the general solution is

$$v(t) = \frac{g}{k} + Ke^{-kt} \ , \tag{6.81}$$

for some constant k.

Example 6.23 Suppose a heavy spherical object is throuwn from anairplane at 10000 meters, and that the coefficient of resistance of air is $k = 0.05$. Find the velocity as a function of time. What is the free fallvelocity? Approximately how long does it take to reach the ground?

Here $g = 9.8$, so the free fall velocity is $v_p = 9.8/(.05) = 196$ meters/sec. The general solution to the problem is

$$v(t) = 196 + Ke^{-(.02)t} . \tag{6.82}$$

At $t = 0$, $v = 0$, so $0 = 196 + K$, and our solution is

$$v(t) = 196(1 - e^{-(.02)t}) . \tag{6.83}$$

To answer the last question, we have to find distance fallen as a function of time, by integrating the above:

$$x(t) = 196(t + 50e^{-(.02)t}) + C . \tag{6.84}$$

At $t = 0$, $x = 0$; this gives $C = -196(50)$, and the solution for our particular object:

$$x(t) = 196(t + 50(e^{-(.02)t} - 1)) . \tag{6.85}$$

Now we want to solve for t when $x = 10,000$. For large t, theexponential term is negligible, so T, the time to reach ground, is approximately given by the solution of

$$10,000 = 196(T - 50) \tag{6.86}$$

or $T = 101$ seconds.

§6.5. Inverse Functions

The functions e^x and $\ln x$ are inverses to each other in the sense that the two statements

$$y = e^x \qquad x = \ln y \tag{6.87}$$

are equivalent. In general, two functions f, g are said to be inverse to each other when the statements

$$y = f(x) \qquad x = g(y) \tag{6.88}$$

are equivalent for x in the domain of f, and y in the domain of g. Often we write $g = f^{-1}$ and $f = g^{-1}$ to express this relation. Another way of giving this criterion is

$$f(g(x)) = x \qquad g(f(x)) = x . \tag{6.89}$$

We have to be careful, in discussing inverses, to clearly indicate the domain and range. For example, x^2 and $\sqrt{x}$ appear to be inverses since $(\sqrt{x})^2 = x$. But $\sqrt{x^2}$ is not necessarily x, for x might have been negative: $\sqrt{(-5)^2} = 5$. This is clarified by restricting the domains of both functions to the positive numbers, and by adopting the convention that $\sqrt{x}$ always means the positive square root. That is, we specify the domain and range of $\sqrt{x}$; having done so, it is the inverse to x^2: the statements

$$y = x^2 \qquad x = \sqrt{y} \tag{6.90}$$

Figure 6.5

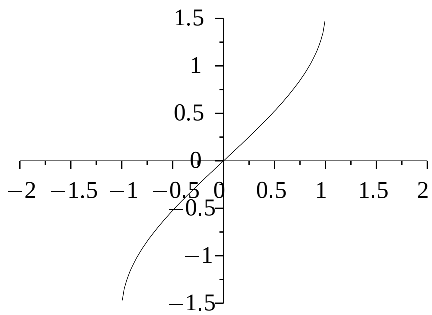
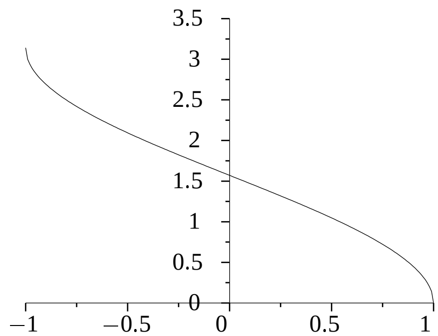


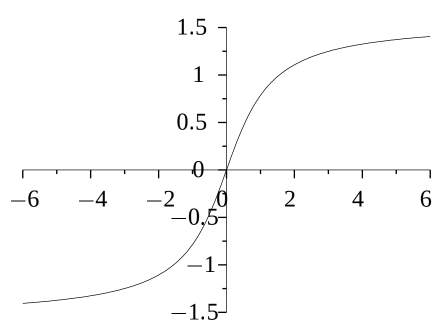
Figure 6.6



are equivalent for the domains $x \geq 0,\ y \geq 0$.

Now, we want to define inverses to the trigonometric functions. Consider first the sine function. Since it is periodic, the equation $\sin y = x$ has many solutions for x between -1 and 1. But, if we insist that y be between $-\pi/2$ and $\pi/2$, there is only one solution. So, to pick a definite inverse for the sine function, we specify that its domain is the interval $[-1,1]$, and its range (set of values) is $[-\pi/2,\pi/2]$. Then, with this specification, it is true that the equation $\sin y = x$ has one and only one solution. That solution we call the *inverse sine function*, denoted $\arcsin x$. To repeat: $\arcsin x$, for x between -1 and 1 is the angle between $-\pi/2$ and $\pi/2$ whose sine is x. See figure 6.5 for a graph of $y = \arcsin x$. Since $\cos(-x) = \cos(x)$, it is not possible to define an inverse if we take the domain of cos to be any interval about 0. However, we note that since the cosine function is strictly decreasing between 0 and π, we can define an inverse on the interval $[-1,1]$ taking values between 0 and π: this is the *inverse cosine*, denoted $\arccos x$. (See figure 6.6 for the graph). Finally, to define an inverse to the tangent function, we note that it is strictly increasing on the interval $(\pi/2,\pi/2)$ and takes every value between $-\infty$ and ∞. Thus we can define the inverse function for all real numbers, taking values in $(\pi/2,\pi/2)$. (See figure 6.7 for the graph of $y = \arctan x$).

Figure 6.7: $y = \arctan x$



Definition 6.5

a) $y = \arcsin x$ *is defined by the condition* $x = \sin y$ *on the interval* $[-1,1]$, *taking values in* $[-\pi/2,\pi/2]$.

b) $y = \arccos x$ *is defined by the condition* $x = \cos y$ *on the interval* $[-1,1]$, *taking values in* $[0,\pi]$.
c) $y = \arctan x$ *is defined by the condition* $x = \tan y$ *on the interval* $(-\infty,\infty)$, *taking values in* $(-\pi/2,\pi/2)$.

Proposition 6.13 *On their domains of definition, we have these formulae:*

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C \tag{6.91}$$

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}dx = -\arccos x + C \tag{6.92}$$

$$\frac{d}{dx}\arctan x = \frac{1}{(1+x^2)} \qquad \int \frac{1}{(1+x^2)}dx = \arctan x + C \tag{6.93}$$

To verify these differentiation formulae, we implicitly differentiate the defining equation, and then use the appropriate trigonometric identity. For example, from $x = \sin y$, we get $dx = \cos y dy$. But, from the Pythagorean theorem $\cos y = \sqrt{1-(\sin y)^2} = \sqrt{1-x^2}$. Thus $dx = \sqrt{1-x^2}dy$, which is the same as a). We derive b) in the same way. Finally, from $x = \tan y$, we get $dx = (\sec y)^2 dy$, and c) follows from the identity $(\sec y)^2 = 1 + (\tan y)^2$.

Note that, for any acute angle α, its complementary angle is $\pi/2 - \alpha$, thus $\sin\alpha = \cos(\pi/2-\alpha)$. Thus $\arcsin x = \pi/2 - \arccos x$, explaining the coincidence in formulae a) and b).

Example 6.24 Integrate $\sec^2/(\tan^2+1)$.

Make the substitution $u = \tan x,\ du = \sec^2 x dx$. Then

$$\int \frac{\sec^2}{\tan^2+1}dx = \int \frac{du}{u^2+1} = \arctan u + C = \arctan(\tan x) + C = x + C \tag{6.94}$$

which, of course could have been more easily derived using the trigonometric identity $\sec^2 x = 1 + \tan^2 x$.

These ideas can be applied in general to relate the derivatives of functions inverse to each other.

$$y = f(x) \quad \text{if and only if} \quad x = g(y)\ . \tag{6.95}$$

Proposition 6.14 *Suppose that* f *and* g *are inverse to each other in their respective domains. Suppose that* $y = g(x)$. *Then* $g'(x) = 1/f'(y)$.

To see this, differentiate the relations $x = f(y),\ y = g(x)$ implicitly with respect to x:

$$1 = f'(y)\frac{dy}{dx}\ , \quad \frac{dy}{dx} = g'(x)\ , \tag{6.96}$$

so

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(y)}\ . \tag{6.97}$$

Example 6.25 Suppose that g is the inverse to the function

(6.98) $$f(x) = x^2 - 3x - 5$$

for $0 < x < 3/2$. Find $g'(1)$.

The point here is that we don't have to use the quadratic formula to answer this question. Let $y = g(x)$. Since g is inverse to f, we have $x = y^2 - 3y - 5$. First we find the value of y corresponding to $x = 1:\ 1 = y^2 - 3y - 5$ has the solutions $-1, 4$. But since f is restricted to positive values, we must have $4 = g(1)$. Now $f'(y) = 2x - 3$, so

(6.99) $$g'(1) = \frac{1}{f'(4)} = \frac{1}{2(4) - 3} = \frac{1}{5}\ .$$