X. Numerical Methods

10.1 Taylor Approximation

Suppose that f is a function defined in a neighborhood of a point c, and suppose that f has derivatives of all orders near c. In section 5 of chapter 9 we introduced the Taylor polynomials for f:

Definition 10.1. The *Taylor polynomial of degree* n of f, centered at c is

$$(T_c^{(n)}f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \ .$$

We saw, in section 9.5, that the Taylor polynomial of degree n is for the best approximation near c to f by a nth degree polynomial. We recall that fact:

Proposition 10.1. The Taylor polynomial $T_c^{(n)}f$ is the polynomial of degree n or less that approximates f near c to nth order.

This statement is not very useful without some estimate of the error bar in this approximation. This was stated in the preceding chapter without proof as:

Proposition 10.2. Suppose that f is differentiable to order $n+1$ in the interval $[c-a, c+a]$ centered at the point c. Then the *error* in approximating f in this interval by its Taylor polynomial of degree n, $T_c^{(n)}f$ is bounded by

$$\frac{M_{n+1}}{(n+1)!}|x-c|^{n+1} \ , \tag{10.1}$$

where M_{n+1} is a bound of the values of $f^{(n+1)}$ over the interval $[c-a, c+a]$. To be precise, we have the inequality

$$|f(x) - T_c^n f(x)| \leq \frac{M_{n+1}}{(n+1)!}|x-c|^{n+1} \quad \text{for all } x \text{ between } c-a \text{ and } c+a \ .$$

Before demonstrating how to use this estimate, let us see how it comes about. To simplify the notation, we shall take c to be the origin. First, a preliminary step:

Lemma. Suppose that $f(0) = 0, f'(0) = 0, \ldots, f^{(n)}(0) = 0$. Then

$$|f(x)| \leq \frac{M_{n+1}}{(n+1)!}|x|^{n+1} \ .$$

First we show the case $n = 1$. We have $|f''(s)| \leq M_2$ for all $s,\ 0 \leq s \leq x$. So, for any $t,\ 0 \leq t \leq x$, we have

$$|f'(t)| = |\int_0^t f''(s)ds| \leq \int_0^t |f''(s)|ds \leq M_2 \int_0^t ds \leq M_2 t \ .$$

But now,

$$|f(x)| = |\int_0^x f'(t)dt| \leq \int_0^x |f'(t)|dt \leq \int_0^x M_2 t dt \leq M_2 \frac{x^2}{2} \ .$$

Of course, the same argument works for x negative, we just have to be careful with the signs.

Now, we show that if we assume the lemma for $n-1$ we can show it for n, and then invoke the principle of mathematical induction. Suppose we have gotten to the $(n-1)$th case. Then the lemma applies (at $n-1$) to the derivative f'; so we know that

$$|f'(t)| \le \frac{M_{n+1}}{n!}|t|^n \quad \text{for all } t \text{ in the interval } [-a,a] \ .$$

(We have M_{n+1} because the nth derivative of f' is the $(n+1)$th derivative of f). Now we argue independently on each side of 0: for $x > 0$:

$$|f(x)| = |\int_0^x f'(t)dt| \le \int_0^x |f'(t)|dt \le \int_0^x \frac{M_{n+1}}{n!}t^n dt \le \frac{M_{n+1}}{n!}\frac{t^{n+1}}{n+1} = \frac{M_{n+1}}{(n+1)!}|x|^{n+1} \ .$$

The argument for $x < 0$ is the same; just be careful with signs.

Now that the lemma is verified, we go to the proposition itself. Let $g = f - T_c^{(n)}f$ Then g satisfies the hypotheses of the lemma. Furthermore, since $T_c^{(n)}f$ is a polynomial of degree n, its $(n+1)$th derivative is identically zero. Thus $g^{(n+1)}$ has the same bound, M_{n+1}. Applying the lemma to g, we have the desired result:

$$|f(x) - T_c^{(n)}f| \le \frac{M_{n+1}}{(n+1)!}|x|^{n+1} \ .$$

If this error estimate converges to 0 as $n \to \infty$, then we saw that f is be represented by its *Taylor series*:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

in the interval $[c-a, c+a]$.

Before doing some examples, let's review what has to be done. To use the Taylor polynomials to find an approximation to the value to a function within some error bound, we first have to find bounds M_n for the successive derivatives of the function. Then we have to calculate the values of the error estimate

$$E(n) = \frac{M_n}{n!}|x-c|^n \tag{10.2}$$

for successive values of n until we have found one which is within the desired error. Then we calculate the approximation using the Taylor polynomial of degree $n-1$.

Example 10.1. Find $\sqrt{e}$ to within an error of 10^{-4}.

This is $e^{1/2}$, so we look at the function $f(x) = e^x$. Since $f^{(n)}(x) = e^x$ for all n, and the value $x = 1/2$ is within 1 of 0, we can use the Maclaurin series for e^x and the bounds $M_n = e^1$. Since 3 is more manageable than e, we take $M_n = 3$. Now we estimate the error (10.2) at stage n which we'll call $E(n)$. We have, in this example

$$E(n) = \frac{3}{n!}(\frac{1}{2})^n \ .$$

$$n = 1: \quad E(1) = \frac{3}{2}$$

$$n = 2: \quad E(2) = \frac{3}{2}\frac{1}{4}$$

$$n = 3: \quad E(3) = \frac{3}{6}\frac{1}{8} = \frac{3}{48}$$

$$n = 4: \quad E(4) = \frac{3}{24}\frac{1}{16} = \frac{3}{384}$$

$$n = 5: \quad E(5) = \frac{3}{120}\frac{1}{32} = 7.8 \times 10^{-4}$$

$$n = 6: \quad E(6) = \frac{3}{720}\frac{1}{64} < 10^{-4}$$

Thus, we have our estimate to within 10^{-4} by taking the fifth Taylor polynomial:

$$T_0^5(e^x)(1/2) = 1 + \frac{1}{2} + \frac{1}{2}\frac{1}{4} + \frac{1}{6}\frac{1}{8} + \frac{1}{24}\frac{1}{16} + \frac{1}{120}\frac{1}{32} = 1.6487$$

Example 10.2. Find $\sin(\pi/8)$ to within an error of 10^{-3}.

Here we start with the Maclaurin series for $\sin x$:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$$

Since the derivatives of $\sin x$ alternate between $\pm \sin x$ and $\pm \cos x$, we may take $M_n = 1$ for all n. We need to choose n large enough that the Taylor error estimate $E(n)$ satisfies

$$E(n) = \frac{1}{n!}(\frac{\pi}{8})^n < \frac{1}{2}10^{-3} .$$

$$n = 1: \quad E(1) = \frac{\pi}{8} = .3927$$

$$n = 2: \quad E(2) = \frac{1}{2}(\frac{\pi}{8})^2 = .077$$

$$n = 3: \quad E(3) = \frac{1}{6}(\frac{\pi}{8})^3 = .010$$

$$n = 4: \quad E(4) = \frac{1}{24}(\frac{\pi}{8})^4 = .0009$$

so we need only go to $n = 3$. The estimate is

$$T_0^3(\sin x)(\pi/8) = \frac{\pi}{8} - \frac{1}{6}(\frac{\pi}{8})^2 = .3826$$

or $\sin(\pi/8) = .383$, correct to three decimal places.

Some Taylor series converge too slowly to get a reasonable approximation by just a few terms of the series. As a rule, if the series has a factorial in the denominator, this technique will work efficiently, otherwise, it will not.

Example 10.3. Use the Maclaurin series

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

to estimate $\ln(1+a)$ for $|a| < 1/2$ to within 4 decimal places.

First we calculate the successive derivatives of $f(x) = \ln(1+x)$ to obtain the bounds M_n. We have

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$$

The maximum in the specified interval is the value at $x = -1/2$: $(n-1)!2^n$. Thus the error is bounded by

$$E(n) = \frac{(n-1)!}{n!} x^n = \frac{(2|x|)^n}{n} \ .$$

For $|x|$ near $1/2$, this error grows too slowly: for accuracy to four decimal places, we need 10^4 terms. But, say, for $|x| < 1/4$ we'll have $E(n) \leq (2^n n)^{-1}$, which is less than 10^{-4} for $n = 12$, and for $|x| < 1/20$, three terms will suffice.

Let's close this section with a more direct argument for the Taylor estimate, proposition 10.2. Again we assume that f is differentiable of all orders in the interval $[c-R, c+R]$, and that M_n is an upper bound of $|f^{(n)}(x)|$ on that interval. We have to treat the cases $x \leq c$ and $x \geq c$ separately. Here we demonstrate the proposition for x in the interval $[c, c+R]$, the case of the left half interval is the same, but with more care given to signs. First, we show the case $n = 0$. By the fundamental theorem of the calculus,

$$f(x) - f(c) = \int_c^x f'(t)dt \ .$$

Since $f'(t) \leq M_1$ in that interval,

$$f(x) - f(c) \leq \int_c^x M_1 dt = M_1(x-c) \ .$$

We proceed now to the induction step: assume the theorem is true for all functions f and the integer $n-1$. Apply the theorem to the derivative f' of f and the integer $n-1$. Now, for every k, the kth derivative of f' is the $(k+1)$st derivative of f, so, in particular, the bound on the nth derivative of f' is M_{n+1}. The induction hypothesis is this:

$$f'(x) - \big(f'(c) + f''(c)(x-c) + \frac{f^{(3)}(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n))}(c)}{(n-1)!}(x-c)^{n-1}\big) \leq \frac{M_{n+1}}{n!}(x-c)^n \ .$$

Integrate this inequality from c to x, obtaining

$$f(x) - f(c) - \big(f\text{"}(c)(x-c) + f\text{"}(c)\frac{(x-c)^2}{2} + \frac{f^{(3)}(c)}{2!}\frac{(x-c)^3}{3} + \cdots + \frac{f^{(n)}(c)}{(n-1)!}\frac{(x-c)^n}{n}\big)$$

$$\leq \frac{M_{n+1}}{n!}\frac{(x-c)^{n+1}}{n+1} \ ,$$

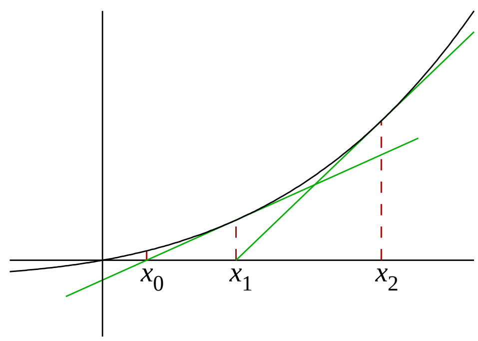
which is the desired result.

10.2 Newton's method

Suppose that we want to estimate the solution of the equation $f(x) = 0$. If this is a linear equation, there is no problem: we just look for the point at which the line crosses the x-axis. Newton's idea is that, since the tangent line approximates the graph, why not take as the estimate the point at which the tangent line crosses the x-axis? Well, to make sense of this, let's start at some value x_0 and calculate $y_0 = f(x_0)$. If that is 0, then we're through. If not, let x_1 be the point at which the tangent line at (x_0, y_0) crosses the x-axis. That is our first approximation. If it is not good enough, replace x_0 with x_1, and repeat the process, over and over again, until the result is good enough. Of course, the definition of "good enough" is to be determined by the degree of precision desired in the context of the problem at hand.

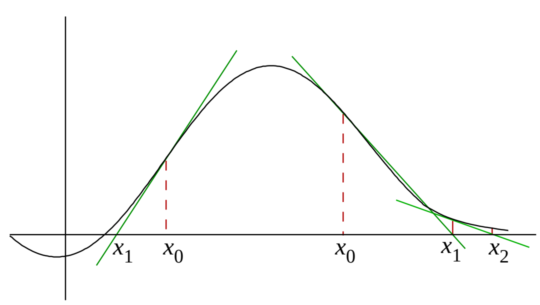
This process is illustrated in figure 10.1.

Figure 10.1



However, it doesn't always work so well, as figure 10.2 shows. The choice of x_0 on the left works, but the choice on the right sets in motin a search for a root that doesn't exist. The important point is to start with a decent guess for x_0, so that we start in a range of the function where the concavity of the curve forces convergence of these successive approximations.

Figure 10.2



Newton's method thus, is a technique for replacing an approximation by a better one. Suppose we start with the function $y = f(x)$, and have found an approximation $x = a$, with $f(a)$ (relatively) close to zero. The slope of the tangent line at $x = a$ is $f'(a)$, and the equation of the tangent line is $y - f(a) = f'(a)(x - a)$. This intersects the x-axis where $y = 0$, so we have

$$(10.3) \qquad -f(a) = f'(a)(x-a) \quad \text{which has the solution} \quad x = a - \frac{f(a)}{f'(a)}$$

We now replace a by this value, and repeat the process. That is, we define a sequence of approximate solutions (hopefully converging to the root), using (10.3) as the recursion relation:

Newton's Method. Given the differentiable function $y = f(x)$, define a sequence recursively as follows:

$$a_0 = \quad \text{a good guess of the solution of} \quad f(x) = 0 \ ,$$

$$(10.4) \qquad a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} \ .$$

If the guess is in an interval containing a root, and $f'(x) > 0$ or $f'(x) < 0$ in that interval, then the sequence a_n converges to a solution of $f(x) = 0$.

The criterion for convergence seems problematic, since we don't know the root of the equation - we're trying to find it. But, ordinarily the relation $y = f(x)$ is one we can graph, and from the graph we can see if the condition is satisfied, and make a good guess. How do we know when we are within an error e of the solution? This is a difficult question, and will be discussed in a course in Numerical Analysis. For our purposes, in the presence of the criterion, we can stop as soon as successive values of a_n are within e of each other.

Example 10.4. Find $\sqrt{8}$ correct to within 4 decimal places.

Here we want to find the root of the equation $f(x) = x^2 - 8 = 0$. Since $f'(x) = 2x$, the criterion is satisfied so long as $x > 0$ Let's start with $x_0 = 3$, since $3^2 = 9$ and 9 is close to 8. The next estimate is

$$x_1 = 3 - \frac{3^2 - 8}{2 \times 3} = 2.8333$$

We now use this in the recursion, and continue until we reach stability in the first 4 decimal places:

$$x_2 = 2.8333 - \frac{(2.8333)^2 - 8}{2 \cdot 2.8333} = 2.8284$$

$$x_3 = 2.8284 - \frac{(2.8284)^2 - 8}{2 \cdot 2.8284} = 2.8284 \ ,$$

which we take as the estimate accurate to 4 decimal places.

Before going on to other examples, we summarize the process: To solve $f(x) = 0$:
1. Graph $y = f(x)$ to find plausible intervals in which to work.
2. Calculate $f'(x)$, and determine an interval $[a.b]$ in which $f'(x)$ does not change sign, but for which the signs of $f(a)$ and $f(b)$ differ.

3.. Select a first estimate a_0 in $[a, b]$ so that $f(a_0)$ is small..
4. Calculate the recursion relation

$$x' = x - \frac{f(x)}{f'(x)} .$$

5. Find a_1 from a_0 by taking $x = a_0,\ x' = a_1$ in the recursion relation.
6. Repeat step 5 until the successive estimates are no further from each other than the desired estimate.

Of course, in practice this is all done on the computer. Furthermore, the way the function is defined may make it difficult, even impossible, to complete some of the steps. In such a case, pick a starting point as best you can and calculate a number of terms. If they don't seem to converge, try another starting point.

Example 10.5. Find the solutions of $f(x) = x^3 - 12x + 1 = 0$ to three decimal places.

First we graph the function so as to make an intelligent first estimate.

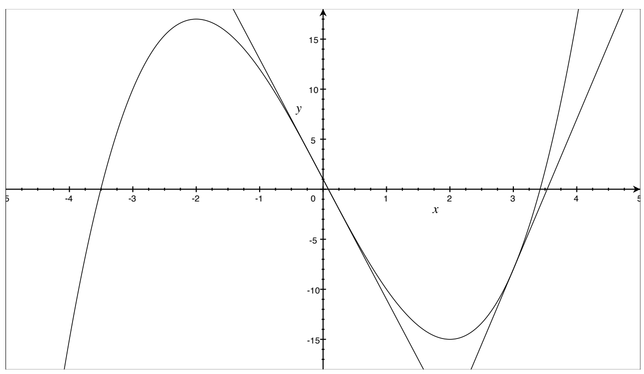


Figure 10.3

We have $f'(x) = 3x^2 - 12$, so the local maximum and minimum of $y = f(x)$ are at (-2,17), (2, -15). At $x = 0$, $y = 1$, and the graph is as shown in figure 10.3. There are three solutions; one close to 0, another larger than 2, and a third less than -2

First, find the solution near zero. We see from the graph that the derivative is well away from zero in a range including $x = 0$ and the solution, so we can take our first estimate to be $x_0 = 0$. Now we calculate successive estimates:

$$x_1 = 0 - \frac{1}{-12} = \frac{1}{12} = .08333$$

$$x_2 = .08333 - \frac{(.08333)^3 - 12(.08333) + 1}{3(.08333)^2 - 12} = .08338 ,$$

$$x_3 = .08338$$

so this solution is $x = .08338$ up to four decimal places.

Now, to find the solution larger than 2, it will not do to take $x_0 = 2$, since the derivative is 0 there. But if we take $x_0 = 3$, we have $f'(3) = 15$, a nice large number, so the recursion should work. We find

$$x_1 = 3 - \frac{3^3 - 12(3) + 1}{3(3^2) - 12} = 3 - \frac{-8}{15} = 3.5333 ,$$

$$x_2 = 3.5333 - \frac{(3.5333)^3 - 12(3.5333) + 1}{3(3.5333)^2 - 12} = 3.4267$$

$$x_3 = 3.4215 \ , \qquad x_4 = 3.4215 \ ,$$

so this is our estimate to four decimal places. In the same way, starting at $x_0 = -3$, we find the third solution.

Example 10.6. Solve $e^x = x + 2$ to three decimal places.

Here $f(x) = e^x - x - 2$, $f'(x) = e^x - 1$. Since the derivative is increasing, and greater than 1 at $x = 1$, and $f(1) < 0$, while $f(2) > 0$, a good first estimate will be any number between 1 and 2. So, take $x_0 = 1$. The recursion is

$$x' = x - \frac{e^x - x - 2}{e^x - 1} = \frac{e^x(x-1) + 2}{e^x - 1} \ .$$

We now calculate the successive estimates:

$$x_1 = 1.16395 \ , \quad x_2 = 1.1464 \ , \quad x_3 = 1.1462 \ , \quad x_4 = 1.1462 \ ,$$

so this is the desired estimate. Notice that in this range, the derivative is not very large, so that the convergence is slower than in the preceding examples.

10.3. Numerical Integration

We have learned techniques for calculating definite integrals which are based on finding antiderivatives of the function to be integrated. However, in many cases we cannot find an expression for the antiderivative, and these techniques will not lead to an answer. For example $f(x) = \sqrt{1 + x^3}$. No formula for the integral exists in any integral tables. In such a case, we have to return to the definition of the integral, and approximate the definite integral by the approximating sums. To explain this, we first review the definition of the definite integral.

10.4 Definition. Let $y = f(x)$ be a function defined on the interval $[a, b]$. The *definite integral* is defined as follows. A *partition* of the interval is any increasing sequence

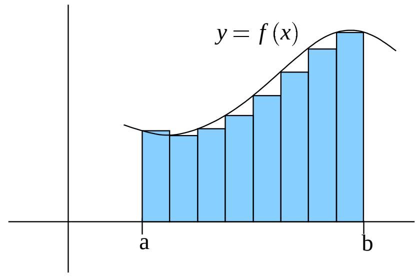
$$\{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

of points in the interval. The corresponding *approximating sum* is

$$\sum_1^n f(x_i')\Delta x_i \qquad (10.5)$$

where Δx_i is the length $x_i - x_{i-1}$ of the ith interval, x_i' is any point on that interval, and $\sum$ indicates that we add all these products together (see Figure 10.4).

Figure 10.4: Approximation to the area under a curve.



If these approximating sums approach a limit as the partition becomes increasingly fine (the lengths of the subdivisions go to zero), this limit is the *definite integral* of f over the interval $[a, b]$, denoted

$$\int_a^b f(x)dx \; .$$

Thus, we can approximate a definite integral by the sums (10.5). One way to accomplish this is: Pick an integer N, and divide the interval $[a, b]$ into N subintervals, all of size $(b - a)/N$. For each subinterval, evaluate the function at the right endpoint x_i, and form the sum

$$\sum_1^N f(x_i)\Delta x_i = \frac{(b-a)}{N}\sum_1^N f(x_i) \quad \text{(Approximating Rectangles)} \tag{10.6}$$

Example 10.7. Let's find an approximate value for $\int_0^1 \sqrt{1 + x^3}dx$ this way. Let's divide the interval into 10 subintervals. Then the sum (7) is

$$\frac{1}{10}(\sqrt{1 + (1/10)^3} + \sqrt{1 + (2/10)^3} + \sqrt{1 + (3/10)^3} + \cdots \sqrt{1 + (10/10)^3}) =$$

$$= \frac{1}{10}(1.0005 + 1.0040 + 1.0134 + 1.0315 + 1.0607 + 1.1027 + 1.1589 + 1.2296 + 1.3149 + 1.4142) =$$

$$= 1.1330 \; .$$

Of course these calculations are tedious if done by hand, but, by computer - completely trivial. It is a good idea to try these using a spreadsheet, because there you get to follow the computation. If we take more subdivisions, we get a better approximation. For example, if we take $N = 100$ we get

$$\frac{1}{100}(\sqrt{1 + (1/100)^3} + \sqrt{1 + (2/100)^3} + \sqrt{1 + (3/100)^3} + \cdots \sqrt{1 + (100/100)^3}) = 1.113528 \; ,$$

an apparently better approximation.

Example 10.8. Just to see how well this method is working, let us use it to approximate $\int_1^2 x^{1.4}dx$, which we know to be $(2^{2.4} - 1)/(2.4) = .1.782513\ldots$. Let's first take $N = 10$. The approximating sum is

$$\frac{1}{10}((1.1)^{1.4} + (1.2)^{1.4} + (1.3)^{1.4} + \cdots (2)^{1.4})$$

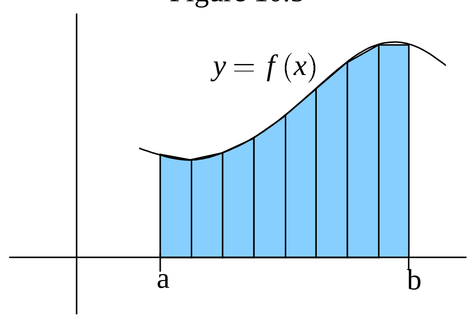
$$= \frac{1}{10}(1.1427 + 1.2908 + 1.4438 + 1.6017 + 1.7641 + 1.9309 + 2.1020 + 2.2771 + 2.4562 + 2.6390)$$

$$= 1.8648 \ .$$

For $N = 100$ we obtain the estimate 1.790712, which is better, but not great.

We can improve this method by improving the estimate in each subinterval. First, note that we have estimated the integral in each subinterval by the area of the rectangle of height at the right endpoint. If instead we estimate this area using the trapezoid whose upper side is the line segment joining the two endpoints (see figure 10.5), it makes sense that this is a better estimate.

Figure 10.5



This comes down to

(10.7) $$\frac{(b-a)}{2N}(f(a) + 2\sum_{1}^{N-1} f(x_i) + f(b)) \quad \text{(Trapezoid Rule)}$$

Going one step further, we might replace the upper curve by the best parabolic approximation. For N even, this leads to the rule

(10.8) $$\frac{(b-a)}{3N}(f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{N-1}) + f(b)) \quad \text{(Simpson's Rule)}$$

Let us now do the above examples using these two rules:

Example 10.9. The calculation of $\int_0^1 \sqrt{1+x^3}dx$ using the Trapezoidal rule and $N = 10$ gives us

$$\frac{1}{20}(1 + 2\big(\sqrt{1+(1/10)^3} + \sqrt{1+(2/10)^3} + \sqrt{1+(3/10)^3} + \cdots \sqrt{1+(9/10)^3}\big) + \sqrt{2}) =$$

$$= \frac{1}{20}(1 + 2\big(1.0005 + 1.0040 + 1.0134 + \cdots + +1.2296 + 1.3149\big) + 1.4142) =$$

$$= 1.0123 \ .$$

Example 10.10. Let's compare the estimates of $\int_0^1 x^{1.4}dx$ with these two new methods.. First, with $N = 10$, and the trapezoid rule:

$$\frac{1}{20}(1 + 2\big((1.1)^{1.4} + (1.2)^{1.4} + (1.3)^{1.4} + \cdots (1.9)^{1.4}\big) + 2^{1.4} =$$

$$= \frac{1}{20}\big(1 + 2(1.1427 + 1.2908 + \cdots + 2.2771 + 2.4562\big) + 2.6390) =$$

$$= 1.78288 \quad \text{(Trapezoid Rule)} \; .$$

Now Simpson's rule and $N = 10$:

$$= \frac{1}{30}(1 + 4(1.1427) + 2(1.2908) + \cdots + 2(2.2771) + 4(2.4562) + 2.6390)$$

$$= 1.782513 \quad \text{(Simpson's Rule)} \; .$$

In general, these estimate of a definite integral get better as N gets larger, and the trapezoid rule is better than the rectangular sums, but not as good as Simpson's rule, simply because the local approximations to the curve are better. The question is, of course: how good are these rules: what is the error for a given N and a given rule? The following proposition gives the estimates (these are not easy to derive).

10.5 Proposition. Let f be a function defined on the interval $[a, b]$, and let M_n be a bound on the nth derivative of f in this interval. Then, the approximations to $\int_a^b f(x)dx$ using N subdivisions are correct to within the error $E(N)$ given by:

$$E(N) = \frac{(b-a)^3}{12N^2} M_2 \quad \text{(Trapezoid Rule)} \; ; \qquad E(N) = \frac{(b-a)^5}{180N^4} M_4 \quad \text{(Simpson's Rule)} \; .$$

Example 10.11. Let us calculate the error in the trapezoid estimate for $\int_0^1 \sqrt{1+x^3}dx$ given in example 10.9. We first have to find a bound on the second derivative. Differentiating twice, we have

$$f''(x) = \frac{3(2x + \frac{x^4}{2})}{2(1+x^3)^{3/2}} \; ,$$

which is bounded in [0,1] by 15/4. Since $N = 10$, the error is less than

$$\frac{1^3}{12(10^2)} \frac{15}{4} = .003125,$$

so the answer 1.0123 of example 10.9 is correct to two decimals.

Example 10.12. Now, let's calculate the error in the use of Simpson's rule in example 10, with $N = 10$. First we need to bound the fourth derivative of $f(x) = x^{1.4}$ in the interval [1,2]. A calculation leads to

$$|f^{(4)}(x)| = |(-1.6)(-.6)(.4)(1.4)x^{-2.6}| \leq .5376$$

Thus the error is bounded by

$$\frac{1^5}{180(10)^4}(.5376) < 3 \times 10^{-8} \; ,$$

telling us that the calculation of example 10.10 is correct in all six decimal places.

Problems, Chapter 10.

1. Suppose that we want to find values of $\ln(1 + a)$ for $a > 0$. The value of M_n found in example 10.3 involves negative numbers. But, according to the proof of the error estimate for Taylor series

at the end of section 10.1, we need only find the maximum of the nth derivative in the interval between 0 and a. Show that this gives much faster convergence of the Taylor series in the interval between 0 and 1/2 than found in example 10.3.

2. Since $\tan(\pi/6) = 1/\sqrt{3}$,and therefore, $\pi = 6 \arctan(1/\sqrt{3})$ we can use the Taylor series for the arc tangent to estimate π. Do this, using the first three nonzero terms.

3. Since $\sin(\pi/6) = 1/2$, we can also find π by solving the equation $\sin x = 1/2$. We can approximate the solution by replacing sin by an approximating Taylor polynomial, and then using Newton's method. Do this with the cubic Taylor polynomial for $\sin x$.

4. Find a solution, by Newton's method, of the equation

$$x^5 - x^4 + x^3 - x^2 = 4$$

correct to five decimal places.

5. Here is another way of estimating π. We know that

$$\pi/4 = \int_0^1 \frac{dx}{1+x^2} \ .$$

Estimate this integral by the trapezoid rule, using steps of size 1/10. How many steps should we take to be sure of an estimate correct to 4 decimal places?

6. Define

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)(n+1)!} \ .$$

Evaluate $J_0(1)$ correctly to 4 decimal places.

7. Find an estimate for

$$\int_0^2 \frac{\sin x}{x} dx$$

using Simpson' s rule with $N = 20$ subdivisions.