

XI. Conics and Polar Coordinates

11.1 Quadratic Relations

A quadratic relation between the variables x , y is an equation of the form

$$(11.1) \quad Ax^2 + By^2 + Cxy + Dx + Ey = F \quad \text{so long as one of } A, B, C \text{ is not zero .}$$

If we substitute a number for x , we obtain a quadratic equation in y , which we can then solve by the quadratic formula. In this way (11.1) defines y as a function of x implicitly, although for some x there may be two solutions or no solutions. In any event, the set of points in the x, y -plane satisfying equation (11.1) is a curve. These curves are called *conics* or *conic sections*, for they represent, in suitably chosen coordinates, the curve on a cone in three dimensions cut out by a plane.

As a first example, consider the equation

$$(11.2) \quad x^2 + y^2 = F .$$

This is the circle of radius \sqrt{F} when $F > 0$, just the origin if $F = 0$, and has no points if $F < 0$.

We will see that in general, the curve defined by the quadratic relation (11.1) is one of these three curves: a) parabola, b) ellipse, c) hyperbola. As we have seen in (11.2), for special values of the coefficients, there may be no curve. There are other possibilities; for example, the equation $x^2 - y^2 = a$ describes a hyperbola if $a \neq 0$, but if $a = 0$, we get the two lines $x = \pm y$.

First we list the *standard forms* of the basic curves. These are standard in the sense that any other curve given by a quadratic equation is obtained from one of these by moving the curve in the plane by translating and/or rotating.

The Parabola. The standard form is one of these:

$$(11.3) \quad y = ax^2, \quad x = ay^2$$

The sign of a determines the orientation of the parabola. This gives us four possibilities, the graphs of which are shown in figures 11.1-11.4.

Figure 11.1

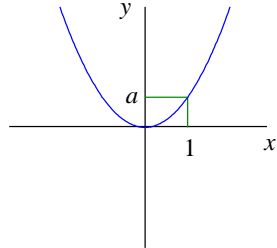


Figure 11.2

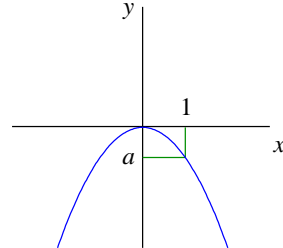


Figure 11.3

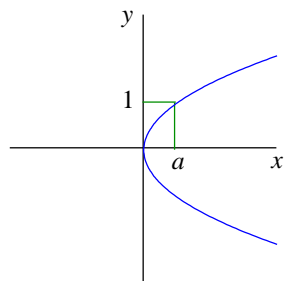
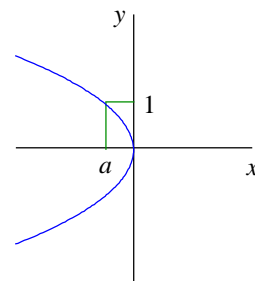


Figure 11.4



The magnitude of a determines the spread of the parabola: for $|a|$ very small, the curve is narrow,

and as $|a|$ gets large, the parabola broadens. The origin is the *vertex* of the parabola. In the first two cases, the y -axis is the *axis* of the parabola, in the second two cases it is the x -axis. The parabola is symmetric about its axis.

The Ellipse. The standard form is

$$(11.4) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The values x can take lie between $-a$ and a and the values of y lie between $-b$ and b .

If $a > b$ (as shown in figure 11.5), the *major axis* of the ellipse is the x -axis, the *minor axis* is the y -axis and the points $(\pm a, 0)$ are its *vertices*. If $a < b$ (as shown in figure 11.6), the major axis of the ellipse is the y -axis, the x -axis is the minor axis, and the points $(0, \pm b)$ are its *vertices*.

Of course, if $a = b$, the curve is the circle of radius $|a|$, and there are no special vertices or axes.

Figure 11.5

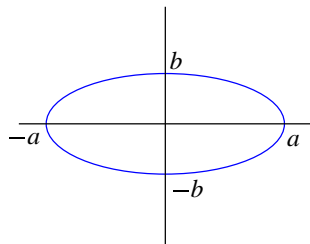
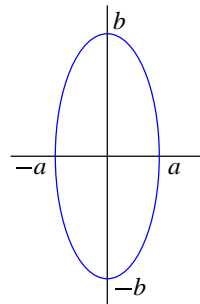


Figure 11.6



The Hyperbola. The standard form is one of these:

$$(11.5) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 ,$$

corresponding to the graphs (11.7),(11.8) respectively.

The x -axis is the *axis* of the hyperbola (11.7). The points $(\pm a, 0)$ are the *vertices* of the hyperbola; for x between these values, there corresponds no point on the curve. Similarly for the hyperbola (11.8), the y -axis is its *axis* and the points $(0, \pm b)$ are its *vertices*.

The lines

$$y = \pm \frac{b}{a}x$$

are the *asymptotes* of the hyperbola, in the sense that, as $x \rightarrow \infty$, the curve gets closer and closer to these lines. We see this by multiplying the defining equation by b^2/x^2 , and consider what happens as $x \rightarrow \infty$. For example, using the first equation of (11.5), we get

$$\frac{b^2}{a^2} - \frac{y^2}{x^2} = \frac{b^2}{x^2} \quad \text{or}$$

$$\frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2} .$$

Thus, as $|x|$ gets large, the hyperbola approaches the graph of

$$\frac{y^2}{x^2} = \frac{b^2}{a^2}$$

which amounts to the two equations $y = \pm(b/a)x$. The dotted lines of figures 11.7 and 11.8 represent the asymptotes.

Figure 11.7

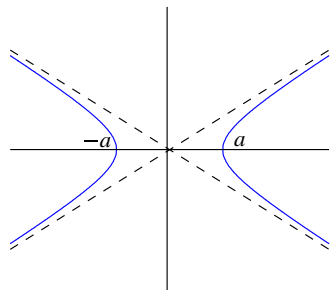
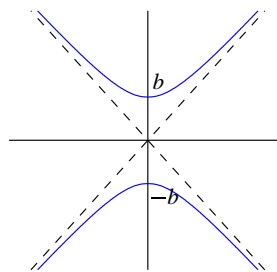


Figure 11.8



Now, let's return to the general quadratic relation (11.1), with no xy term, $C = 0$:

$$Ax^2 + By^2 + Dx + Ey + F = 0 ,$$

and see how to relate the equation to the above standard forms. By completing the square in both x and y we are led to an equation which looks much like one of the standard forms, but with the

center removed to a new point (x_0, y_0) . If $C \neq 0$, the situation is more difficult: a rotation of the figure is also required to get it into standard form. We leave that for a later discussion, and here consider only the case $C = 0$. First, some examples:

Example 11.1. Let's graph the curve

$$3x^2 - 30x - y + 73 = 0$$

We have to complete the square in x . We get

$$3(x^2 - 10x + 25) - y + 73 - 75 = 0$$

which can be rewritten in the standard form

$$y + 2 = 3(x - 5)^2 ,$$

where the vertex is at $(5, -2)$ rather than the origin (see figure 11.10).

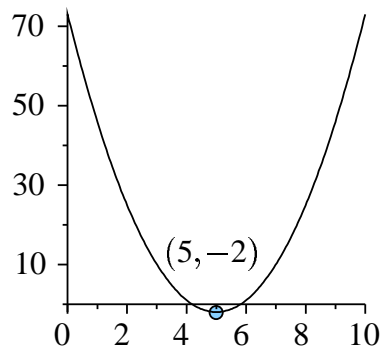


Figure 11.10

Example 11.2. Graph the curve

$$9x^2 + 4y^2 - 18x - 16y = 11$$

Completing the squares:

$$9(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = 11 + 9 + 16 = 36 \quad \text{or}$$

$$9(x - 1)^2 + 4(y - 2)^2 = 36$$

which can be rewritten in standard form (with the point $(1, 2)$ replacing the origin):

$$\frac{(x - 1)^2}{2^2} + \frac{(y - 2)^2}{3^2} = 1 .$$

See figure 11.11.

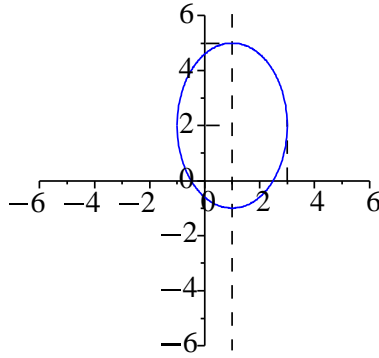


Figure 11.11

Example 11.3. Graph the curve

$$-5x^2 + y^2 + 30x + 4y - 46 = 0$$

Completing the squares:

$$-5(x^2 - 6x + 9) + (y^2 + 4y + 4) = 46 - 45 + 4 = 5 ,$$

$$\frac{(y + 2)^2}{(\sqrt{5})^2} - (x - 3)^2 = 1 .$$

This is a hyperbola with center at the point (3,-2) and axis the line $x = 3$.

Proposition 11.1. The equation

$$Ax^2 + By^2 + Dx + Ey = F$$

can be put into one of the following forms by completing the square:

a) (parabola) : $y - y_0 = A(x - x_0)^2, \quad \text{if } B = 0 .$

The vertex of the parabola is at (x_0, y_0) , and the axis is the line $x = x_0$.

b) (parabola) : $x - x_0 = C(y - y_0)^2 \quad \text{if } A = 0 .$

The vertex of the parabola is at (x_0, y_0) , and the axis is the line $y = y_0$.

c) (ellipse) : $\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1 \quad \text{if } A \text{ and } B \text{ are of the same sign .}$

The center of the ellipse is at (x_0, y_0) , and its axes are the lines $x = x_0, y = y_0$.

d) (hyperbola) : $\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1 \quad \text{or} \quad \frac{(y - y_0)^2}{b^2} - \frac{(x - x_0)^2}{a^2} = 1$

if A and B are of different signs. The center of the hyperbola is (x_0, y_0) , and its axes are the lines $x = x_0, y = y_0$.

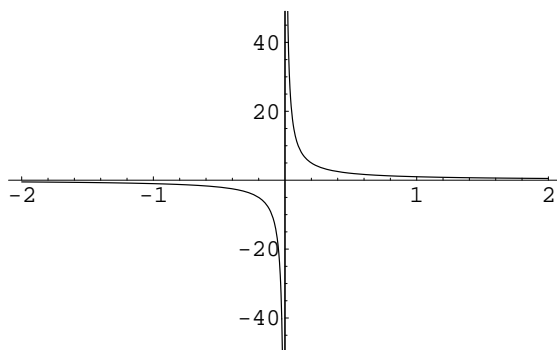
e) If both A and B are zero, the curve is a line. The following degenerate cases may also result:

$$A(x - x_0)^2 + B(y - y_0)^2 \leq 0 : \quad \text{no graph or just the point } (x_0, y_0).$$

$$A(x - x_0)^2 - B(y - y_0)^2 = 0 : \quad \text{two lines crossing at } (x_0, y_0).$$

Example 11.4. Finally, just to illustrate the situation of a quadratic whose coefficient of xy is nonzero, we consider the curve $xy = 1$. This curve is symmetric about the lines $y = \pm x$, and has the asymptotes $x = 0$, $y = 0$. This appears to be a hyperbola with major axis the line $x = y$. In fact, if we make the linear change of variables $x = u + v$, $y = u - v$, this becomes the curve $u^2 - v^2 = 1$ in the new variables. (This change of variables represents a rotation by 45° , with a slight change of scale.) See figure (11.12).

Figure 11.12



Problems 11.1

In problems 1-4 put the conic in standard form, and find the center and vertices. In the next section you will learn about foci of the conic sections. After reading that section, return to these problems and find the foci.

1. $y - 8x^2 + 32x - 29 = 0$

2. $9x^2 + 4y^2 - 36x + 8y + 4 = 0$

3. $4x^2 - y^2 + 2y = 5$

4. $x^2 - 5y^2 - 4x + 10y = 1$

5. Find the equation of the parabola with vertex $(2,5)$, axis $y = 5$ that goes through the point $(4,2)$.

6. Find the equation of the ellipse with vertices $(0,0)$, $(0,10)$ that goes through the point $(4,6)$.

7. Find the equation of the hyperbola with vertices at $(3,0)$, $(1,0)$ and asymptotes of slope ± 5 .

In problems 8-10 you are given the equation of a curve C and a point P on the curve. Find the point of intersection of the tangent line to C at P with the x -axis.

8. $x^2 + 5y = 0$, $(10, -20)$.

9. $x^2 + 4y^2 = 16$, $(2\sqrt{3}, 1)$.

10. $4x^2 - y^2 = 1$, $(\sqrt{2}/2, 1)$.

11.2 Eccentricity and Foci

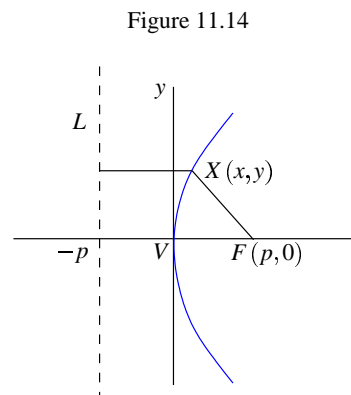
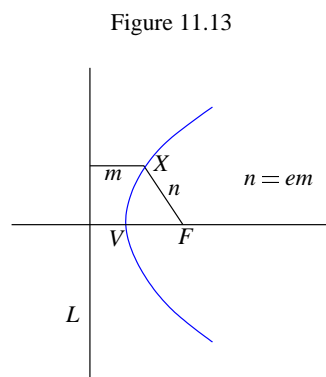
Curves described by quadratic equations are called the *conic sections* because they can be visualized as the intersection of a cone with a plane. We shall now consider another definition, dating from the ancient Greeks, which leads to important properties of the conics.

Fix a point F and a line L in the plane such that L does not go through F .

Pick a positive number e . We consider the locus C of all points X in the plane such that

$$(11.6) \quad |XF| = e|XL|$$

where $|XY|$ means the distance from X to Y , whether Y is a point or a line. e is the *eccentricity* of C ; F the *focus* and L the *directrix*. Note that the curve C is symmetric about the line through the focus and perpendicular to the directrix. This is the *axis* of the curve. There is one point between F and L on C which is on this axis; this point is the *vertex* of C . See figure 11.13.



We now show that if $e = 1$, C is a parabola, if $e < 1$, C is an ellipse and if $e > 1$, C is a hyperbola. Let's take the axis of C to be the x axis, and place the vertex at the origin, O . Then the focus is some point $(p, 0)$; we take $p > 0$. Since $|OF| = p$, from (11.6) we find that the directrix is the line $x = -p/e$ (see figure 11.14).

Now, for a point $X = (x, y)$ on the curve, we have

$$|XL| = x + p/e \quad \text{and} \quad |XF| = \sqrt{(x - p)^2 + y^2}$$

and so equation (11.6) in coordinates is given by

$$(11.7) \quad \sqrt{(x - p)^2 + y^2} = e(x + p/e) = ex + p .$$

Case $e = 1$. Squaring both sides we get

$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2 \quad \text{simplifying to} \quad y^2 = 4px .$$

This of course is the standard form of a parabola. It also locates the focus (at $(p, 0)$) and the directrix (the line $x = -p$) of the parabola.

Proposition 11.2 . The focus of the parabola $y^2 = ax$ is $a/4$ units on one side of the vertex of the parabola along the axis, and the directrix intersects the axis $a/4$ units on the other side.

Example 11.5. Find the vertex, focus and directrix of the parabola given by the equation

$$2x^2 + 12x - y + 20 = 0$$

We put the equation in standard form. First we move the terms not involving x to the other side:

$$2x^2 + 12x = y - 20 ,$$

and then, completing the square, we have

$$2(x^2 + 6x + 9) = y - 20 + 18 , \quad \text{or} \quad (x + 3)^2 = \frac{1}{2}(y - 4) .$$

Thus the vertex is at $(-3, 4)$, axis of the parabola is the line $y = 4$ and the parabola opens up to the right (since $4p = 1/2 > 0$). We have $p = 1/8$, so the focus is $1/8 = .125$ units above the vertex at $(-3, 4.125)$ and the directrix is the line $y = 3.875$.

Example 11.6. Find the equation of the parabola whose vertex is at $(4, 2)$ and whose directrix is the line $x = -1$. Find the focus of this parabola.

Since the directrix is a vertical line, the axis is horizontal, so the equation has the form

$$(y - 2)^2 = 4p(x - 4) ,$$

since the vertex is at $(4, 2)$. Now p is the distance between the vertex and the directrix, so $p = 4 - (-1) = 5$. Thus the equation of the parabola is

$$(y - 2)^2 = 20(x - 4) .$$

The focus is 5 units to the right of the vertex, so is at (9, 2).

Example 11.7. Find the equation of the parabola whose focus is the origin and whose vertex is at the point $(a, 0)$ with $a > 0$.

Since both the focus and vertex are on the x -axis, that is the axis of the parabola. Since the vertex is to the right of the focus, the parabola opens to the left. Thus the equation has the form

$$y^2 = -4p(x - a) ,$$

where p is the distance between focus and vertex. But that is a , so the equation is

$$y^2 = -4a(x - a) .$$

Case $e \neq 1$. Squaring both sides of (11.7) gives us

$$x^2 - 2px + p^2 + y^2 = e^2(x^2 + 2px + p^2)$$

which simplifies to

$$(11.8) \quad (1 - e^2)x^2 + y^2 - 2p(1 + e)x = 0$$

Thus, if $e < 1$, this is an ellipse, and if $e > 1$ this is a hyperbola. Notice, because of symmetry in the minor axis, ellipses and hyperbolas have two foci; one on each side of the minor axis.

We now want show how to locate the foci of an ellipse ($e < 1$) given in standard form. Thus we start by putting (11.8) in standard form, and then compare it to the formula of Proposition 11.1c). Dividing equation (11.8) by the coefficient of x^2 gives us

$$x^2 - \frac{2p}{1 - e}x + \frac{y^2}{1 - e^2} = 0$$

Now completing the square, we come to

$$(11.9) \quad \left(x - \frac{p}{1 - e}\right)^2 + \frac{y^2}{1 - e^2} = \frac{p^2}{(1 - e)^2}$$

Comparing this to

$$(11.10) \quad \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

we see that the center of the ellipse is at $(p/(1 - e), 0)$ and $a^2 = p^2/(1 - e)^2$, $b^2 = a^2(1 - e^2)$. Let c be the distance of the center from the focus. Since the focus is at $(p, 0)$,

$$c = \frac{p}{1 - e} - p = p\left(\frac{1}{1 - e} - 1\right) = e\frac{p}{1 - e} = ea$$

and $c^2 = e^2 a^2 = a^2 - b^2$. Summarizing

Proposition 11.3. If an ellipse is in standard form

$$(11.10) \quad \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1 ,$$

with $a > b$, then the foci of the ellipse are on the major axis, c units away from the center where

$$c^2 = a^2 - b^2$$

The eccentricity of the ellipse is given by the equations

$$b^2 = (1 - e^2)a^2 \quad \text{or} \quad e = c/a$$

A similar argument for the case $e > 1$, the hyperbola, leads to

Proposition 11.4 . If a hyperbola is in standard form

$$(11.11) \quad \frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1$$

then the foci of the hyperbola are on the major axis, c units away from the center where

$$c^2 = a^2 + b^2$$

The eccentricity of the hyperbola is given by the equations

$$b^2 = (e^2 - 1)a^2 \quad \text{or} \quad e = c/a$$

Example 11.8. Find the foci of the conic given by the equation

$$x^2 + 4y^2 - 2x = 8 .$$

First, we complete the square to get the equation in standard form:

$$\frac{(x - 1)^2}{3^2} + \frac{y^2}{(3/2)^2} = 1 .$$

This conic is an ellipse centered at $(1,0)$, with major axis the line $x = 1$, and $a^2 = 9$, $b^2 = 9/4$. Thus $c^2 = a^2 - b^2 = 9(3/4)$, so $c = (3/2)\sqrt{3}$. This is the distance of the foci from the center (along the major axis), so the foci are at $(1 \pm (3/2)\sqrt{3}, 0)$.

Example 11.9. Find the foci of the conic given by the equation

$$y^2 - x^2 + 4x = 13 .$$

Complete the squares, and get the standard form

$$\frac{y^2}{3^2} - \frac{(x - 2)^2}{3^2} = 1 .$$

This is a hyperbola with center at $(2, 0)$, and major axis the line $x = 2$. We have $c^2 = a^2 + b^2 = 18$, so $c = 3\sqrt{2}$ is the distance of the foci from the center along the line $x = 2$. Thus the foci are at $(2, \pm 3\sqrt{2})$. The vertices are at $(2, \pm 3)$.

Example 11.10. Find the equation of the ellipse centered at the origin, with a focus at $(2, 0)$ and a vertex at $(3, 0)$.

The equation of an ellipse centered at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

we are given $a = 3$, $c = 2$. Thus $b^2 = a^2 - c^2 = 5$, and the equation is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

Problems 11.2.

1. Find the equation of the parabola whose vertex is at $(0, 2)$ and focus is the origin.
2. Find the vertex of the parabola with focus at $(0, 7)$ and passes through the points $(\pm 2, 7)$ and $(\pm 1, 5)$.
3. Find the equation of the parabola with focus at $(2, 3)$ and directrix the line $y = -1$.
4. Find the foci and eccentricity of the ellipse given by the equation

$$\frac{(x - 1)^2}{4} + \frac{y^2}{16} = 1 .$$

5. A hyperbola has vertices at $(\pm 3, 0)$ and foci at $(\pm 4, 0)$. What are the equations of its asymptotes?
6. Find the foci and eccentricity of the hyperbola

$$\frac{(y - 2)^2}{25} - \frac{(x + 1)^2}{16} = 1 .$$

7. A hyperbola has asymptotes $y - 1 = \pm 0.8(x + 2)$. What is its eccentricity?

In each of problems 8 through 10, the curve described depends upon a parameter. Identify the parameter, and find the equation of the curve in terms of the parameter.

8. A parabola with axis the x -axis and focus at the origin.
9. A hyperbola with foci at $(-1, 0)$, $(1, 0)$.
10. An ellipse with foci at $(-1, 0)$, $(1, 0)$.
11. Show that the hyperbola and the ellipse of problems 9 and 10 intersect orthogonally; that is, at a point of intersection their tangent lines are orthogonal.

11.3 String and Optical Properties of the Conics

We have seen that the parabola can be defined as the locus of points X equidistant from a given point F and a given line L . The ellipse and the hyperbola have similar definitions.

Proposition 11.5. Given two points F_1 and F_2 and a number a greater than half the distance between F_1 and F_2 , the locus of points X such that

$$(11.13) \quad |XF_1| + |XF_2| = 2a$$

is an ellipse with foci at F_1 and F_2 and major axis of length $2a$.

Choose coordinates so that the points F_1 and F_2 lie on the x -axis, equidistant from the origin. Then F_1 has coordinates $(-c, 0)$, and F_2 has coordinates $(c, 0)$ for some $c < a$. Let X have the coordinates (x, y) . Then (11.13) becomes

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

Eliminate the radicals to verify that we end up with a quadratic equation which is that of an ellipse.

We should point out that every ellipse has property (11.13). For there is only one ellipse with given foci and length $2a$ of the major axis. So, if we start with a given ellipse, and then construct the curve satisfying (11.13) with the foci and major axis of the given ellipse, since that curve is an ellipse, it is the given ellipse.

We have a similar description of the hyperbola:

Proposition 11.6. Given two points F_1 and F_2 and a positive number a , the locus of points X such that

$$(11.13) \quad |XF_1| - |XF_2| = 2a$$

is a hyperbola with foci at F_1 and F_2 .

Actually, this is just the branch of the hyperbola which wraps around the focus F_2 ; the other branch is given by the equation

$$|XF_2| - |XF_1| = 2a$$

Figure 11.15

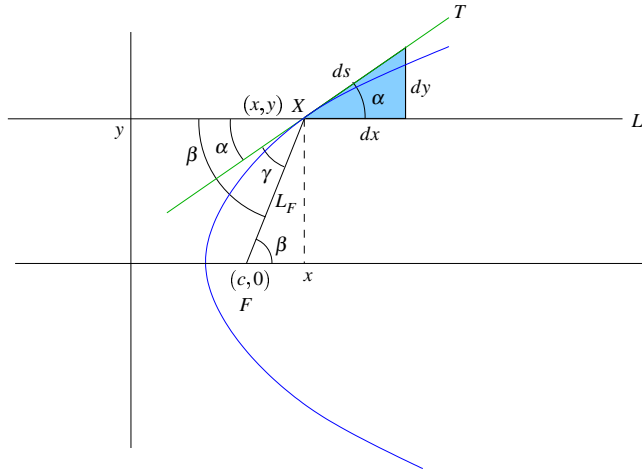
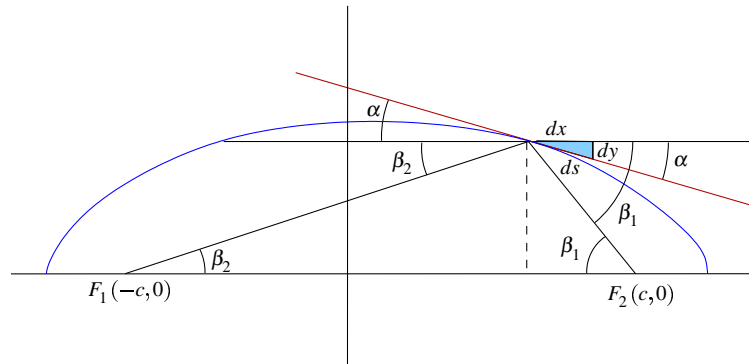


Figure 11.16



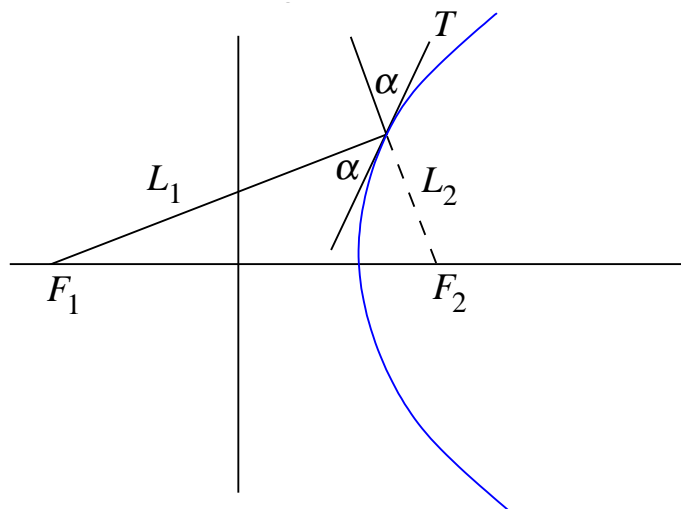


Figure 11.17

The optical properties of the conics follow from these string characterizations. Let's start with the parabola. Suppose that the parabola is coated with a light-reflecting material. The rays of a beam of light originating far away along the axis of the parabola will approach the parabola along lines parallel to its axis. According to the physics of the situation, the angle of reflection off the parabola is equal to the angle of incidence. The optical property of the parabola is that these reflected rays all meet at the focus. See figure 11.15.

Proposition 11.7. Let X be a point on the parabola, and T the tangent line to the parabola at X . Let L_F be the line from the focus to X , and L the line through X parallel to the axis of the parabola. Then the angle between T and L_F is equal to the angle between T and L .

What we want to show, referring to figure 11.15, is that $\gamma = \alpha$. From the figure we see that $\gamma = \beta - \alpha$, so this amounts to showing that $\beta - \alpha = \alpha$.

Referring to figure 11.15, the focus-directrix definition of the parabola tells us that L_F (the distance from the point X to the focus F) is equal to $x + c$ (the distance from X to the directrix). Thus

$$(11.15) \quad L_F = x + c .$$

Squaring this equation gives us

$$(x - c)^2 + y^2 = (x + c)^2 .$$

Differentiate this equation with respect to arc length and divide by 2 to get

$$(x - c) \frac{dx}{ds} + y \frac{dy}{ds} = (x + c) \frac{dx}{ds} .$$

Now, divide by $x + c = L_F$ to get

$$(11.16) \quad \frac{x - c}{L_F} \frac{dx}{ds} + \frac{y}{L_F} \frac{dy}{ds} = \frac{dx}{ds} .$$

From figure 11.15, we have

$$\frac{dx}{ds} = \cos \alpha, \quad \frac{dy}{ds} = \sin \alpha, \quad \frac{x-c}{L_F} = \sin \beta, \quad \frac{y}{L_F} = \cos \beta,$$

so (11.16) becomes

$$\sin \beta \cos \alpha + \cos \beta \sin \alpha = \cos \alpha, \quad \text{or} \quad \cos(\beta - \alpha) = \cos \alpha,$$

from which we conclude that $\beta - \alpha = \alpha$, as desired.

The optical property of the ellipse is that a ray of light emanating from one focus reflects off the ellipse so as to pass through the other focus (see figure 11.16).

Proposition 11.8. Let X be a point on the ellipse, and T the tangent line to the ellipse at X . Let L_1 be the line from the focus F_1 to X , and L_2 the line from the other focus F_2 to X . Then the angle between T and L_1 is equal to the angle between T and L_2 .

What we want to show, referring to figure 11.16, is that $\beta_2 + \alpha = \beta_1 - \alpha$. We start with the string property, written in the coordinates as shown in the figure:

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

Differentiate with respect to arc length to arrive at

$$\frac{x+c}{\sqrt{(x+c)^2 + y^2}} \frac{dx}{ds} + \frac{y}{\sqrt{(x+c)^2 + y^2}} \frac{dy}{ds} + \frac{x-c}{\sqrt{(x-c)^2 + y^2}} \frac{dx}{ds} + \frac{y}{\sqrt{(x-c)^2 + y^2}} \frac{dy}{ds} = 0$$

Now make substitutions of the trigonometric functions, using the figure. We have to be careful: in our picture dy and $x-c$ are negative, so since the sine and cosine are ratios of *lengths*, we have

$$\cos \beta_1 = -\frac{x-c}{\sqrt{(x-c)^2 + y^2}} \quad \sin \alpha = \left| \frac{dy}{ds} \right| = -\frac{dy}{ds}.$$

Thus our equation becomes

$$\cos \beta_2 \cos \alpha + \sin \beta_2 (-\sin \alpha) + (-\cos \beta_1) \cos \alpha + \sin \beta_1 (-\sin \alpha) = 0$$

or

$$(\cos \beta_2 \cos \alpha - \sin \beta_2 \sin \alpha) - (\cos \beta_1 \cos \alpha + \sin \beta_1 \sin \alpha) = 0$$

which is $\cos(\beta_2 + \alpha) - \cos(\beta_1 - \alpha) = 0$, so $\beta_2 + \alpha = \beta_1 - \alpha$ as desired.

The optical property of the hyperbola is that a ray of light emanating from one focus reflects off the opposite branch of the hyperbola so as to appear to have come from the other focus (see figure 11.17).

Proposition 11.9. Let X be a point on the hyperbola, and T the tangent line to the ellipse at X . Let L_1 be the line from the focus F_1 to X , and L_2 the line from the other focus F_2 to X . Then the exterior angles between T and L_1 and between T and L_2 are equal.

The verification is a computation completely analogous to the one for the ellipse: differentiate the string property, and then make the correct trigonometric substitutions

Problems 11.3

1. Find the point (x, y) on the parabola $y^2 = 12x$ for which the line from the focus meets the tangent line at an angle of 45° .
2. Give a proof, using the optical properties of the conics that confocal hyperbolae and ellipses intersect orthogonally. That is, suppose that H and E are respectively a hyperbola and an ellipse, and suppose that they have the same foci. Show that, at a point P of intersection of H and E that the curves have tangent lines which are perpendicular.

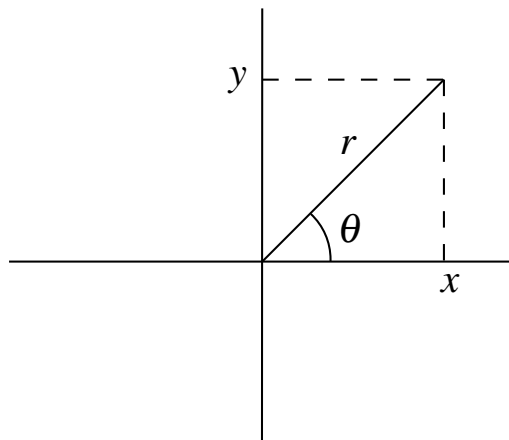
11.4 Polar Coordinates

Often a problem can be seen as that of understanding the motion of a particle \mathbf{P} in the plane relative to a fixed point \mathbf{O} . In such a situation it is desirable to be able to describe the position of \mathbf{P} in terms of the length and the direction of the line between the two points. These are the *polar coordinates* of the point \mathbf{P} . We consider the fixed point \mathbf{O} as the origin of these coordinates, and take the positive x -axis as the “zero” direction. Then any other direction is described by the angle between it and the positive x axis, which we denote as θ . The distance of a point on this line from the origin is denoted r . For a point in the first quadrant, we have these equations relating the cartesian coordinates (x, y) with the polar coordinates r, θ :

$$(11.17) \quad x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{y}{x}$$

See figure 11.18 to justify these formulas. Polar coordinates have two peculiarities that need to be pointed out. Every value of (r, θ) determines a point in the plane. However, if $r = 0$, the point is the origin, and θ doesn't make sense. Secondly, the values (r, θ) and $(r, \theta + 2\pi)$, and in fact, $(r, \theta + 2n\pi)$ for any n give the same point. This ambiguity is sometimes of value: for example, when discussing the motion of a particle, n tells us how many times the particle has wound around the origin in the counterclockwise sense. Finally, it is also of convenience to let r take negative values, meaning a distance of $|r|$ in the opposite direction of the ray θ . Thus (r, θ) and $(-r, \theta + \pi)$ determine the same point. We now consider the graphs of equations in polar coordinates.

Figure 11.18



Example 11.11. The equation $r = a$, for $a > 0$ is satisfied by all points of distance a from the origin, so is polar equation of the circle of radius a centered at the origin.

Example 11.12. The equation $\theta = \theta_0$ is the line which makes an angle of θ_0 with the x -axis.

Example 11.13. $r = a\theta$ describes the motion of a point which rotates around the origin at angular velocity 1 while moving out along the ray at velocity a . This is the **Archimedean spiral**; see figure 11.19.

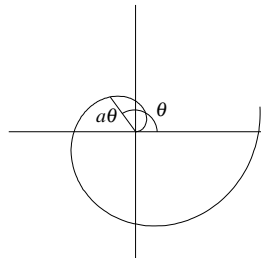


Figure 11.19

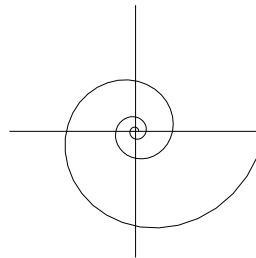


Figure 11.20

Example 11.14. $r = e^{a\theta}$ is another spiral, however, the point moves out along the ray at a rate exponential in the rate of rotation. This is the **logarithmic spiral**, depicted in figure 11.20.

Example 11.15. The equation $r = a \cos \theta$ is a the circle of diameter a with center on the x -axis which goes through the origin. For, if we multiply by r we get $r^2 = ar \cos \theta$, which can now be written in cartesian coordinates (using (11.17)) as

$$x^2 + y^2 = ax \quad \text{or} \quad \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4} .$$

Given an equation of the form $r = r(\theta)$, we can often trace out the graph by just studying the behavior of the function $r(\theta)$. Let's redo example 11.15 this way. We have this table

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
r	a	$\frac{a\sqrt{2}}{2}$	0	$-\frac{a\sqrt{2}}{2}$	$-a$	0	a

It is useful to follow the point on the curve of figure 11.21 as θ ranges from 0 to 2π . Between 0 and $\pi/2$ the point is in the first quadrant, and as the angle increases it moves toward the origin, reaching there at $\theta = \pi/2$. Then for θ between $\pi/2$ and π , the point is in the fourth quadrant

(because $r < 0$), steadily moving away from the origin until we reach the point we've started with. This looks like a circle, and the argument above (in example 11.15) shows that it is. Note that as θ moves from π to 2π the circle is retraced.

Example 11.16. Similarly, the equation $r = a \cos(\theta - \theta_0)$ is the circle through the origin of radius a with center on the ray of angle θ_0 . This amounts to the assertion that any equation of the form

$$r = a \cos \theta + b \sin \theta$$

is a circle with the origin the endpoint of one of its diameters (see problem 1 of this section).

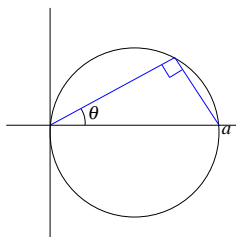


Figure 11.21

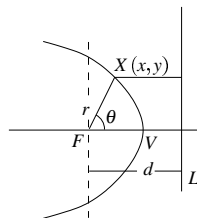


Figure 11.22

Example 11.17. If we are given the equation of a curve in cartesian coordinates, we can find its equation in polar coordinates through the substitution $x = r \cos \theta$, $y = r \sin \theta$. For example

$$(11.18) \text{ Equation of a line : } r = \frac{c}{a \cos \theta + b \sin \theta} .$$

For, the general equation of a line is $ax + by = c$. After substitution this becomes

$$ar \cos \theta + br \sin \theta = c,$$

which gives us (11.18) when we solve for r .

Example 11.18. The polar equation of a conic of eccentricity e , focus at the origin and directrix the line $x = d$ is

$$(11.19) \text{ Equation of a Conic : } r = \frac{ed}{1 + e \cos \theta} .$$

To show (11.19), we start with the defining relation $|XF| = e|XL|$, referring to figure 11.22.

In polar coordinates this gives us

$$r = e(d - x) = e(d - r \cos \theta)$$

Solving for r brings us to (11.19). If the figure is rotated by θ_0 , we just replace θ with $\theta - \theta_0$

Example 11.19. $r = a \cos 2\theta$. We first construct the table:

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	a	0	$-a$	0	a

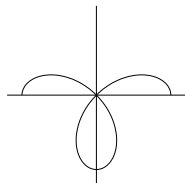


Figure 11.23

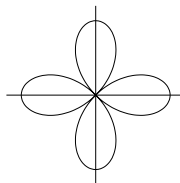


Figure 11.24

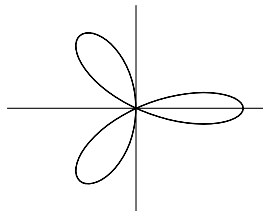


Figure 11.25

Follow this discussion along the graph in figure 11.23. This time the curve starts (at $\theta = 0$) at $r = a$, and decreases to zero by $\theta = \pi/4$. Between $\pi/4$ and $\pi/2$, r is negative, so the curve is in the third quadrant, and as θ rotates counterclockwise, r moves away from the origin finally to $r = -a$ for $\theta = \pi/2$. As θ increases from $\pi/2$ the point continues to move toward the origin (in the fourth quadrant), arriving there at $\theta = 3\pi/2$. Moving on, r becomes positive, so we enter the second quadrant with the distance from the origin steadily increasing until, at $\theta = \pi$ we are at $r = a$. Since $\cos \theta$ is an even function, as we move from π to 2π (or what is the same, from $-\pi$ to 0), we just get the same curve, reflected in the x -axis. The result is the **four-petaled rose** shown here in figure 11.24.

Example 11.20 $r = a \cos 3\theta$ is a three-petalled rose. Construct the table of important values between 0 and π and argue as in example 11.19. The table is

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	a	0	$-a$	0	a	0	$-a$

That completes the rose; of figure 11.25; as we proceed from π to 2π we traverse the rose again.

We conclude

Proposition 11.10. The graph of the equation $r = a \cos(n\theta)$ or $r = a \sin(n\theta)$ is a $2n$ -petalled rose if n is even, and an n petalled rose if n is odd (traversed twice).

Limaçons. These are the curves defined by the equation $r = a + b \cos \theta$.

First, we consider the case: $a > b$. We have the table

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$a + b$	a	$a - b$	a	$a + b$

leading us to figure 11.26.

As b gets closer and closer to a , the value of r for $\theta = \pi$ goes to zero. Thus when $a = b$, we get the graph of figure 11.27, called the **cardioid**.

Then as b goes beyond a , r becomes negative as θ gets near π , and there is an inner loop of the limaçon.

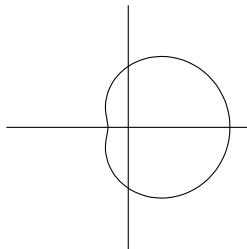


Figure 11.26

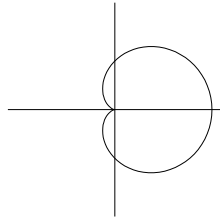


Figure 11.27

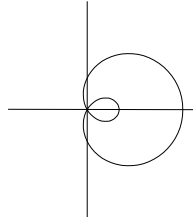


Figure 11.28

Example 11.21. $r = 2 + 4 \cos \theta$. Our table is this:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	6	2	-2	2	6

When $\cos \theta = -1/2$, that is, for $\theta = \pm 2\pi/3$, the value of r is zero, and between these two values r is negative. Following these values, we arrive at the graph in figure 11.28.

We have drawn the curve so that it is tangent to the ray $\theta = \pm \pi/3$ as the moving point comes to the origin. As we shall see in the next section, this is correct.

Finally, it is important to note that if the function $\cos \theta$ is replaced by $-\cos \theta$ the curve is reflected in the y -axis, and if it is replaced by $\pm \sin \theta$, it is rotated by a right angle.

Problems 11.4

1. Show that the graph of the polar equation $r = a \cos \theta + b \sin \theta$ is a circle of radius $\sqrt{a^2 + b^2}$ going through the origin. Where is its center?
2. Graph $r = 3(\cos \theta + \sqrt{3} \sin \theta)$.
3. What is the polar equation of an ellipse, with one focus at the origin, corresponding directrix the line $x = -3$ and corresponding vertex at the point $(-1, 0)$?
4. Identify the curve: $r = 2 \sin(5\theta)$.
5. Graph $r^2 = \cos(2\theta)$. This is called a *lemniscate*.

11.5 Calculus in polar coordinates

Arc length

Consider the curve given in polar coordinates by the equation $r = r(\theta)$. We can calculate the differential ds of arc length by the differential triangle in polar coordinates using the diagram in figure 11.29 (where we have written t for θ).

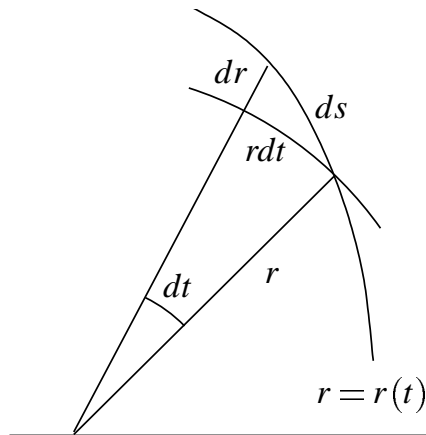


Figure 11.29

The length of the arc of the circle of radius r subtended by the angle $d\theta$ is $r d\theta$. The differential triangle is thus a right triangle with side lengths dr and $r d\theta$. By the pythagorean theorem

$$(11.20) \quad ds^2 = dr^2 + r^2 d\theta^2$$

Example 11.22. Find the length of the curve $r = \theta^2$ from 0 to 2π .

This curve is a spiral whose distance from the origin increases as the square of the angle. We have $dr = 2\theta d\theta$, so

$$ds^2 = dr^2 + r^2 d\theta^2 = 4\theta^2 d\theta^2 + \theta^4 d\theta^2 = \theta^2(4 + \theta^2) d\theta^2$$

and thus the length is

$$\int_0^{2\pi} ds = \int_0^{2\pi} \theta \sqrt{4 + \theta^2} d\theta = \frac{1}{3} (4 + \theta^2)^{3/2} \Big|_0^{2\pi} = \frac{1}{3} (4 + 4\pi^2)^{3/2} - 4^{3/2}$$

Area

To calculate the area enclosed by a curve given, in polar coordinates, by $r = r(\theta)$, we calculate the differential of area, using figure 11.30

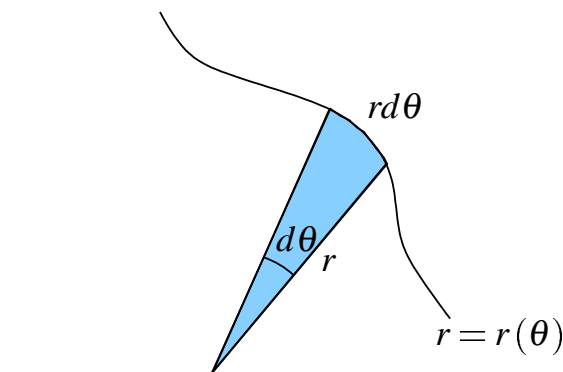


Figure 11.30

The area of the wedge given by the increment $d\theta$ is $(1/2)r^2d\theta$. To see this, we start with the area of the circle of radius r : $A = \pi r^2$. Now an angle α subtends a segment of the circle which is the $(\alpha/2\pi)$ th part of the full circle, thus the area of that segment is $(1/2)r^2\alpha$. Thus, for $\alpha = d\theta$, we get

$$(11.21) \quad dA = \frac{1}{2}r^2d\theta$$

Example 11.23. Find the area enclosed by the cardioid $r = 3(1 + \sin \theta)$.

The area is

$$Area = \frac{1}{2} \int_0^{2\pi} [3(1 + \sin \theta)^2]d\theta = \frac{9}{2} \int_0^{2\pi} (1 + 2 \sin \theta + \sin^2 \theta)d\theta .$$

Now, we know that the integral of $\sin \theta$ over an entire period is zero, so we can neglect the middle term. We now use the double angle formula for the last term, and drop the integral of $\cos(2\theta)$ for the same reason:

$$Area = \frac{9}{2} \int_0^{2\pi} (1 + \frac{1 - \cos(2\theta)}{2})d\theta = \frac{9}{2} \int_0^{2\pi} \frac{3}{2}d\theta = \frac{27}{2}\pi .$$

Example 11.24. Find the area inside one petal of the rose $r = \sin 3\theta$.

At $\theta = 0$ we have $r = 0$, but then as the angle rotates, r increases to its maximum at $3\theta = \pi/2$, and then decreases back to zero for $3\theta = \pi$. Thus one petal is spanned as θ ranges from 0 to $\pi/3$. We now calculate;

$$Area = \frac{1}{2} \int_0^{\pi/3} \sin^2(3\theta)d\theta = \frac{1}{2} \int_0^{\pi/3} (\frac{1 - \cos(6\theta)}{2})d\theta = \frac{1}{2} (\frac{\theta}{2} - \frac{\cos(6\theta)}{12}) \Big|_0^{\pi/3} = \frac{\pi}{12} .$$

Tangents

Given the polar equation $r = r(\theta)$ of a curve, we can find the tangent at any point as follows. First of all, the cartesian coordinates are given by $x = r(\theta) \cos \theta$, $y = r(\theta) \sin \theta$. If m is the slope of the tangent line, we have, by the chain rule

$$(11.22) \quad m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}}$$

Notice that, as $r \rightarrow 0$, the right hand side approaches $\tan \theta$. Thus, if θ_0 is a value for which $r = 0$, then the curve approaches the origin along the ray $\theta = \theta_0$.

Example 11.25. What is the slope of the tangent to the inner loop of the limaçon

$$r = 2 + 5 \cos \theta$$

at the origin?

First, we find the values of θ for which $r = 0$:

$$2 + 5 \cos \theta = 0 \quad \text{or} \quad \cos \theta = -\frac{2}{5}$$

so that $\theta = \pm 0.63\pi$ radians or 113.6° .

Problems 11.5

1. Find the length of the spiral $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$.
2. Find the length of the spiral $r = e^{-\theta}$ for $\theta \geq 0$.
3. Find the area inside the limaçon $r = 3 + \sin \theta$.
4. Find the area inside the cardioid $r = 1 - \sin \theta$ and above the x -axis.
5. What is the slope of the spiral $r = \theta$ at the points $\theta = 2\pi n$ for n a positive integer? What about the spiral $r = e^\theta$ at the same points?
6. Find the tangents to the curve $r = 2 + 3 \sin \theta$ at the origin.