Calculus II
Practice Problems 8: Answers

For each problem, determine whether or not the series converges or diverges. Give your reasoning.

1. \[ \sum_{n=1}^{\infty} \frac{n+1}{n^3} \]

**Answer.** This series converges, by comparison with \( \sum(1/n^2) \):

\[
\frac{n+1}{n^3} \leq \frac{2n}{n^3} \leq \frac{2}{n^2} .
\]

2. \[ \sum_{n=2}^{\infty} \frac{(n+1)^2}{n^3 \ln n} \]

**Answer.** This series looks like \( \sum 1/(n \ln n) \), so diverges:

\[
\frac{(n+1)^2}{n^3 \ln n} \geq \frac{n^2}{n^3 \ln n} \geq \frac{1}{n \ln n} .
\]

3. \[ \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+21} \]

**Answer.** Since \( n/(n+21) \) does not converge to zero as \( n \) goes to infinity, the series cannot converge.

4. \[ \sum_{n=1}^{\infty} \frac{2^n}{n!} \]

**Answer.** Here we’ll use the ratio test:

\[
\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \frac{2}{n+1} \rightarrow 0 ,
\]

so the series converges.

5. \[ \sum_{n=1}^{\infty} \frac{n^e}{e^n} \]

**Answer.** Again, by the ratio test:

\[
\frac{(n+1)^e e^n}{e^{n+1} n^e} = \frac{1}{e} \left( \frac{n+1}{n} \right)^e \rightarrow \frac{1}{e} < 1 ,
\]

so the series converges.

6. \[ \sum_{n=1}^{\infty} \frac{n^{5/2}}{n^4 - n^3 + n^2 + 1} \]
**Answer.** This series converges since the order of the numerator is 5/2, and that of the denominator is 4, and 4 is greater than 5/2 by more than 1.

7. \[ \sum_{n=1}^{\infty} \frac{n!n}{(2n)!} \]

**Answer.** Try the ratio test:

\[
\frac{(n+1)!((n+1) (2n)!)}{n!n} = \frac{n+1}{(2n+2)} \frac{2n+1}{n(2n+2)(2n+1)} = \frac{(n+1)^2}{(2n+2)}(2n+1)n \to 0 ,
\]

since the degree of the denominator is greater than the degree of the numerator. Thus the series converges.

8. \[ \sum_{n=1}^{\infty} \frac{n^2 + 1}{n^n} \]

**Answer.** This converges since the order of the denominator (3+1/2) is more than 1 more than the order of the numerator.

9. \[ \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \]

**Answer.** Now we know that ln \( x \) is of order less than the order of any positive power of \( x \), by l’Hôpital’s rule:

\[
\lim_{x \to \infty} \frac{\ln x}{x^p} = \lim_{x \to \infty} \frac{1/x}{px^{p-1}} = \lim_{x \to \infty} \frac{1}{px^{p-1}} \to 0 ,
\]

since \( p \) is positive. In particular then, eventually \( \ln n \leq n^{1/2} \), so

\[
\frac{\ln n}{n^2} \leq \frac{n^{1/2}}{n^2} \leq \frac{1}{n^{3/2}} ,
\]

so by the comparison test, our series converges.

We could also use the integral test directly:

\[
\int_2^A \frac{\ln x}{x^2}dx = \left( -\frac{\ln x}{x} \right)|_2^A \to \frac{\ln 2}{2} + \frac{1}{2}
\]

as \( A \to \infty \).

10. \[ \sum_{n=1}^{\infty} \frac{2^n n^3}{n!} \]

**Answer.** This converges by the ratio test:

\[
\frac{2^{n+1}(n+1)^3}{(n+1)!} \frac{n!}{2^n n^3} = \left( \frac{n+1}{n} \right)^3 \frac{2}{n+1} \to 0 .
\]
11. For what positive integers $k$ (if any) does the following series converge?

$$\sum_{n-k}^{\infty} \frac{k!(n-k)!}{n!}$$

**Answer.** Trying the ratio test, we look at

$$\frac{k!(n+1-k)!}{(n+1)!} \frac{n!}{k!(n-k)!} = \frac{n-k+1}{n+1} \to 1,$$

so that tells us nothing. So instead we (do what we should have done first) look directly at the $n$th term:

$$\frac{k!(n-k)!}{n!} = \frac{k!}{n(n-1) \cdots (n-k+1)}$$

has degree at least 2 (for $n$ much bigger than $k$), so long as $k > 1$. So the series converges for $k > 1$. 