Calculus II
Practice Problems 8: Answers

For each problem, determine whether or not the series converges or diverges. Give your reasoning.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3}$

Answer. This series converges, by comparison with $\sum(1/n^2)$:

$$\frac{n+1}{n^3} \le \frac{2n}{n^3} \le \frac{2}{n^2} .$$

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2. $\displaystyle\sum_{n=2}^{\infty} \frac{(n+1)^2}{n^3 \ln n}$

Answer. This series looks like $\sum 1/(n\ln n)$, so diverges:

$$\frac{(n+1)^2}{n^3 \ln n} \ge \frac{n^2}{n^3 \ln n} \ge \frac{1}{n \ln n} .$$

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3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+21}$

Answer. Since $n/(n+21)$ does not converge to zero as n goes to infinity, the series cannot converge.

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4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Answer. Here we'll use the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \frac{2}{n+1} \to 0 ,$$

so the series converges.

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5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^e}{e^n}$

Answer. Again, by the ratio test:

$$\frac{(n+1)^e}{e^{n+1}} \frac{e^n}{n^e} = \frac{1}{e} \left(\frac{n+1}{n}\right)^e \to \frac{1}{e} < 1 ,$$

so the series converges.

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6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{5/2}}{n^4 - n^3 + n^2 + 1}$

Answer. This series converges since the order of the numerator is 5/2, and that of the denominator is 4, and 4 is greater than 5/2 by more than 1.

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7. $\sum_{n=1}^{\infty} \frac{n!n}{(2n)!}$

Answer. Try the ratio test:

$$\frac{(n+1)!(n+1)}{(2n+2)!}\frac{(2n)!}{n!n} = \frac{n+1}{n}\frac{n+1}{(2n+2)(2n+1)} = \frac{(n+1)^2}{(2n+2)(2n+1)n} \to 0 ,$$

since the degree of the denominator is greater than the degree of the numerator. Thus the series converges.

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8. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3\sqrt{n}}$

Answer. This converges since the order of the denominator (3+1/2) is more than 1 more than the order of the numerator.

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9. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Answer. Now we know that $\ln x$ is of order less than the order of any positive power of x, by l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\ln x}{x^p} =^{l'H} \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^{1+p-1}} \to 0 ,$$

since p is positive. In particular then, eventually $\ln n \leq n^{1/2}$, so

$$\frac{\ln n}{n^2} \leq \frac{n^{1/2}}{n^2} \leq \frac{1}{n^{3/2}} ,$$

so by the comparison test , our series converges.

We could also use the integral test directly:

$$\int_2^A \frac{\ln x}{x^2}dx = (-\frac{\ln x}{x} - \frac{1}{x})\Big|_2^A \to \frac{\ln 2}{2} + \frac{1}{2}$$

as $A \to \infty$.

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10. $\sum_{n=1}^{\infty} \frac{2^n n^3}{n!}$

Answer. This converges by the ratio test:

$$\frac{2^{n+1}(n+1)^3}{(n+1)!}\frac{n!}{2^n n^3} = (\frac{n+1}{n})^3 \frac{2}{n+1} \to 0 .$$

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11. For what positive integers k (if any) does the following series converge?

$$\sum_{n=k}^{\infty} \frac{k!(n-k)!}{n!}$$

Answer. Trying the ratio test, we look at

$$\frac{k!(n+1-k)!}{(n+1)!} \frac{n!}{k!(n-k)!} = \frac{n-k+1}{n+1} \to 1 ,$$

so that tells us nothing. So instead we (do what we should have done first) look directly at the nth term:

$$\frac{k!(n-k)!}{n!} = \frac{k!}{n(n-1)\cdots(n-k+1)}$$

has degree at least 2 (for n much bigger than k), so long as $k > 1$. So the series converges for $k > 1$.