Calculus II
Practice Problems 6: Answers

Determine whether or not the integral converges. If it does, try to find its value (you may not be able to do this in some cases).

1. $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^2} =$

Answer. This integral converges. We make the substitution $u = \ln x,\ du = dx/x$. We get

$$\int_2^a \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln a} \frac{du}{u^2} = -u{-1}\Big|_{\ln 2}^{\ln a} = \frac{1}{\ln 2} - \frac{1}{\ln a} \to \frac{1}{\ln 2}$$

as $a \to \infty$.

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2. $\displaystyle\int_1^{10} \frac{dx}{x\sqrt{\ln x}} =$

Answer. Since $\ln 1 = 0$, the problem is at the lower limit of integration. Nevertheless, this integral converges. We make the same substitution $u = \ln x$ and get (taking $a > 0$ and small)

$$\int_a^{10} \frac{dx}{x\sqrt{\ln x}} = \int_{\ln a}^{\ln 10} \frac{du}{u^{1/2}} = 2u^{1/2}\Big|_{\ln 2}^{\ln 10} = 2\sqrt{\ln 10} - 2\sqrt{\ln a} \to 2\sqrt{\ln 10}$$

as $a \to 1$.

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3. $\displaystyle\int_{1/5}^\infty \frac{\ln(5x)}{x^2} dx =$

Answer. Since $\ln x < \sqrt{x}$ for x large enough, the integrand is less than $5x^{-3/2}$, so by comparison (propoosition 8.6), our integral converges. We can now proceed to evaluate it. We make the substitution $u = \ln(5x),\ du = dx/x$, and $x = e^{-u}/5$:

$$\int_{1/5}^a \frac{\ln(5x)}{x^2} dx = 5\int_0^{\ln(5a)} e^{-u} u\,du = 5(e^{-u}u - e^{-u})\Big|_0^{\ln(5a)} = 5(\frac{\ln(5a)}{5a} - \frac{1}{5a} + 1)$$

which converges to 5 as $a \to \infty$.

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4. $\displaystyle\int_{-\infty}^\infty \frac{dx}{(1+x^2)^{3/2}} =$

Answer. This integral converges, by the comparison text. For, $(1+x^2)^{3/2} \geq 1 + x^2$, so

$$\frac{1}{(1+x^2)^{3/2}} \leq \frac{1}{1+x^2}\ ,$$

and the integral of the latter is finite ($= \pi$). Now, we can find the value by a trigonometric substitution. Let $x = \tan u,\ dx = \sec^2 u\,du,\ \sqrt{1+x^2} = \sec u$. (Draw the triangle corresponding to these substitutions!). This gives us

$$\int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{\sec^2 u\,du}{\sec^3 u} = \int \cos u\,du = \sin u + C = \frac{x}{\sqrt{1+x^2}} + C\ .$$

Thus

$$\int_{-a}^{b} \frac{dx}{(1+x^2)^{3/2}} = \frac{x}{\sqrt{1+x^2}}\Big|_{-a}^{b} = \frac{b}{\sqrt{1+b^2}} - \frac{-a}{\sqrt{1+a^2}} = \frac{b}{\sqrt{1+b^2}} + \frac{a}{\sqrt{1+a^2}} .$$

Now the limit of this, as a and b go to infinity is (as we saw in problem 10 of practice set 5) 1+1 =2.

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5. $\displaystyle\int_0^{\pi/2} \frac{dx}{1-\cos x} =$

Answer. The improperness here is at $x = 0$ ($\cos(0) = 1$).There is no easy comparison to make, so we calculate the integral. To do so we use some trigonometric identities:

$$\frac{1}{1-\cos x} = \frac{1}{1-\cos x}\frac{1+\cos x}{1+\cos x} = \frac{1+\cos x}{1-\cos^2 x} = \frac{1+\cos x}{\sin^2 x} .$$

Thus

$$\int \frac{dx}{1-\cos x} = \int \csc^2 x dx + \int \cot x \csc x dx = -\cot x - \csc x .$$

Now we calculate

$$\int_a^{\pi/2} \frac{dx}{1-\cos x} = -\cot(\pi/2) - \csc(\pi/2) + (\cot a + \csc a) = -1 + (\cot a + \csc a) .$$

If we let $a \to 0^+$, the term in parentheses becomes infinite. Thus the integral diverges.

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6. $\displaystyle\int_0^1 \frac{dx}{(1-x)^{3/2}} =$

Answer. Make the change of variable $u = (1-x)$, $du = -dx$. Then the integral becomes $\int_0^1 u^{-3/2} du$. But we saw (see display (7) of the Notes), this integral diverges.

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7. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{x}(1-x)}$

Answer. In this range $1-x \geq 1/2$, so

$$\frac{1}{\sqrt{x}(1-x)} \leq \frac{2}{\sqrt{x}} ,$$

so by comparison the integral converges.

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8. Find the area under the curve $y = (x^2-x)^{-1}$, above the x-axis and to the right of the line $x = 2$.
Answer. First, we find the indefinite integral. We find the partial fraction representation

$$\frac{1}{x^2-x} = \frac{1}{x-1} - \frac{1}{x} .$$

$$\int \frac{dx}{x^2-x} = \int \frac{dx}{x-1} - \int \frac{dx}{x} = \ln(\frac{x-1}{x}) .$$

Now,

$$\int_2^a \frac{dx}{x^2-x} = \ln(\frac{x-1}{x})\Big|_2^a = \ln(\frac{a-1}{a}) - \ln(\frac{2-1}{2}) = \ln(\frac{a-1}{a}) + \ln 2 .$$

As $a \to \infty$, the first term goes to zero, so the answer is $\ln 2$.