Determine whether or not the integral converges. If it does, try to find its value (you may not be able to do this in some cases).

1. \( \int_{2}^{\infty} \frac{dx}{x(x \ln x)^2} \)

**Answer.** This integral converges. We make the substitution \( u = \ln x, \ du = dx/x \). We get

\[
\int_{2}^{a} \frac{dx}{x(x \ln x)^2} = \int_{\ln 2}^{\ln a} \frac{du}{u^2} = -u^{-1}\big|_{\ln 2}^{\ln a} = \frac{1}{\ln 2} - \frac{1}{\ln a} \to \frac{1}{\ln 2}
\]

as \( a \to \infty \).

2. \( \int_{1}^{10} \frac{dx}{x \sqrt{\ln x}} \)

**Answer.** Since \( \ln 1 = 0 \), the problem is at the lower limit of integration. Nevertheless, this integral converges. We make the same substitution \( u = \ln x \) and get (taking \( a > 0 \) and small)

\[
\int_{a}^{10} \frac{dx}{x \sqrt{\ln x}} = \int_{\ln a}^{\ln 10} \frac{du}{u^{1/2}} = 2u^{1/2}\big|_{\ln a}^{\ln 10} = 2\sqrt{\ln 10} - 2\sqrt{\ln a} \to 2\sqrt{\ln 10}
\]

as \( a \to 1 \).

3. \( \int_{1/5}^{\infty} \frac{\ln(5x)}{x^2} \, dx = \)

**Answer.** Since \( \ln x < \sqrt{x} \) for \( x \) large enough, the integrand is less than \( 5x^{-3/2} \), so by comparison (proposition 8.6), our integral converges. We can now proceed to evaluate it. We make the substitution \( u = \ln(5x), \ du = dx/x \), and \( x = e^{-u}/5 \):

\[
\int_{1/5}^{a} \frac{\ln(5x)}{x^2} \, dx = 5 \int_{0}^{\ln(5a)} e^{-u} \, du = 5(e^{-u} - e^{-u})\big|_{0}^{\ln(5a)} = 5\left(\frac{\ln(5a)}{5a} - \frac{1}{5a} + 1\right)
\]

which converges to 5 as \( a \to \infty \).

4. \( \int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^{3/2}} = \)

**Answer.** This integral converges, by the comparison text. For, \((1 + x^2)^{3/2} \geq 1 + x^2 \), so

\[
\frac{1}{(1 + x^2)^{3/2}} \leq \frac{1}{1 + x^2},
\]

and the integral of the latter is finite (\( = \pi \)). Now, we can find the value by a trigonometric substitution. Let \( x = \tan u, \ dx = \sec^2 u \, du, \ \sqrt{1 + x^2} = \sec u \). (Draw the triangle corresponding to these substitutions!). This gives us

\[
\int \frac{dx}{(1 + x^2)^{3/2}} = \int \frac{\sec^2 u \, du}{\sec^3 u} = \int \cos u \, du = \sin u + C = \frac{x}{\sqrt{1 + x^2}} + C.
\]
Thus
\[
\int_{-a}^{b} \frac{dx}{(1 + x^2)^{3/2}} = \frac{x}{\sqrt{1 + x^2}} \bigg|_{-a}^{b} = \frac{b}{\sqrt{1 + b^2}} - \frac{-a}{\sqrt{1 + a^2}} = \frac{b}{\sqrt{1 + b^2}} + \frac{a}{\sqrt{1 + a^2}}.
\]
Now the limit of this, as \(a\) and \(b\) go to infinity is (as we saw in problem 10 of practice set 5) \(1 + 1 = 2\).

5. \(\int_{0}^{\pi/2} \frac{dx}{1 - \cos x} = \)

**Answer.** The improperness here is at \(x = 0\) (\(\cos(0) = 1\)). There is no easy comparison to make, so we calculate the integral. To do so we use some trigonometric identities:

\[
\frac{1}{1 - \cos x} = \frac{1}{1 - \cos x} \frac{1 + \cos x}{1 + \cos x} = \frac{1 + \cos x}{1 - \cos^2 x} = \frac{1 + \cos x}{\sin^2 x}.
\]

Thus
\[
\int \frac{dx}{1 - \cos x} = \int \csc x \, dx + \int \cot x \csc x \, dx = -\cot x - \csc x.
\]

Now we calculate
\[
\int_{0}^{\pi/2} \frac{dx}{1 - \cos x} = -\cot(\pi/2) - \csc(\pi/2) + (\cot a + \csc a) = -1 + (\cot a + \csc a).
\]

If we let \(a \to 0^+\), the term in parentheses becomes infinite. Thus the integral diverges.

6. \(\int_{0}^{1} \frac{dx}{(1 - x)^{3/2}} = \)

**Answer.** Make the change of variable \(u = (1 - x)\), \(du = -dx\). Then the integral becomes \(\int_{0}^{1} u^{-3/2} \, du\). But we saw (see display (7) of the Notes), this integral diverges.

7. \(\int_{0}^{1/2} \frac{dx}{\sqrt{1 - x}} = \)

**Answer.** In this range \(1 - x \geq 1/2\), so
\[
\frac{1}{\sqrt{1 - x}} \leq \frac{2}{\sqrt{x}},
\]
so by comparison the integral converges.

8. Find the area under the curve \(y = (x^2 - x)^{-1}\), above the \(x\)-axis and to the right of the line \(x = 2\).

**Answer.** First, we find the indefinite integral. We find the partial fraction representation
\[
\frac{1}{x^2 - x} = \frac{1}{x - 1} - \frac{1}{x}.
\]

Now,
\[
\int_{2}^{a} \frac{dx}{x^2 - x} = \int \frac{dx}{x - 1} - \int \frac{dx}{x} = \ln(\frac{x - 1}{x} \bigg|_{2}^{a} = \ln(\frac{a - 1}{a}) - \ln(2 - 1) = \ln(\frac{a - 1}{a}) + \ln 2.
\]

As \(a \to \infty\), the first term goes to zero, so the answer is \(\ln 2\).