1. Integrate $\int (\ln x)^2\,dx$.

**Answer.** We integrate by parts, using $u = (\ln x)^2$, $du = 2\ln x\,dx$, $dv = dx$, $v = x$:

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx.$$ 

As we saw in example 9 (another integration by parts):

$$\int \ln x\,dx = x\ln x - x + C,$$

so

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2(x\ln x - x) + C. \quad (1)$$

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2. Integrate $\int x^2 \ln x\,dx$.

**Answer.** Let $u = \ln x$, $du = dx/x$, $dv = x^2\,dx$, $v = x^3/3$:

$$\int x^2 \ln x\,dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3}\,dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C.$$

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3. Integrate $\int \arccos x\,dx$.

**Answer.** Let $u = \arccos x$, $du = -dx/\sqrt{1-x^2}$, $dv = dx$, $v = x$:

$$\int \arccos x\,dx = x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,dx. \quad (2)$$

This last we integrate by the substitution $w = 1 - x^2$, $dw = -2xdx$:

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2} \int w^{-1/2}\,dw = -w^{1/2} + C.$$

Putting this back in (2) we obtain

$$\int \arccos x\,dx = x\arccos x - \sqrt{1-x^2} + C.$$

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4. If the region in the first quadrant bounded by the curves $y = 1$, $y = e^x$ and $x = 1$ is rotated about the $y$-axis, what is the volume of the resulting solid?

**Answer.** The region is drawn in figure 1. One can sweep out the volume in the $y$-direction, using the method of washers, or in the $x$ direction, using the method of shells.
Washers. Here, sweeping in the $y$ direction, the differential of volume is $dV = \pi(R^2 - r^2)dy$. The larger radius is $R = 1$, and the smaller radius is $r = \ln y$. Thus

$$Volume = \pi \int_1^e (1 - (\ln y)^2)dy = \pi(y - y(y^2 + 2y\ln y - 2))\bigg|_1^e$$

using the solution to problem 1 above (equation (1)). Then

$$Volume = \pi(e - e + 2e - 2e - (1 - 2)) = \pi .$$

Shells. Now, we sweep out along the $x$-axis, and the differential of volume is $dV = 2\pi rhdx$. The radius is $x$, and the height of the shell is $e^x - 1$. Thus

$$Volume = 2\pi \int_0^1 x(e^x - 1)dx = 2\pi [xe^x - e^x - \frac{x^2}{2}]_0^1$$

using the formula obtained in example 5 of the Notes. We get

$$Volume = 2\pi (e - e - \frac{1}{2} - (0 - 1)) = \pi .$$

5. Integrate $\int \sec^3 x dx$.

**Answer.** First, we use the identity $\sec^2 x = 1 + \tan^2 x$:

$$\int \sec^3 u du = \int (\tan^2 x + 1)\sec x dx = \int \tan^2 x \sec x dx + \int \sec x dx .$$

The last integral was computed in example 3 of the notes, so we concentrate on the first integral. If we write $\tan^2 x \sec x dx = \tan x (\sec x \tan x)dx$, then we can integrate by parts with the substitution $u = \tan x$, $du = \sec^2 x dx$, $dv = \sec x \tan x dx$, $v = \sec x$. Then

$$\int \tan^2 x \sec x dx = \sec x \tan x - \int \sec^3 x dx .$$
It appears we’re back where we started, but not exactly. Substitute this in (3) to get:

\[
\int \sec^3 u \, du = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx.
\]

Moving the second term to the left hand side, and dividing by 2, we have the result

\[
\int \sec^3 u \, du = \frac{1}{2} (\sec x \tan x + \int \sec x \, dx) = \frac{1}{2} (\sec x \tan x) + \frac{1}{4} \ln \left( \frac{1 + \sin x}{1 - \sin x} \right) + C.
\]

Incidentally, the expression found in example 3 of the Notes for \( \int \sec x \, dx \) is not the usual one, but is equivalent to the formula found in most integral tables:

(4) \[ \int \sec x \, dx = \ln |\sec x + \tan x| + C, \]

using this sequence of identities:

\[
\ln \left( \frac{1 + \sin x}{1 - \sin x} \right) = \ln \left( \frac{(1 + \sin x)^2}{1 - \sin^2 x} \right) = \ln \left( \frac{1 + \sin x}{\cos x} \right)^2 \\
= 2 \ln \left| \frac{1 + \sin x}{\cos x} \right| = 2 \ln |\sec x + \tan x|.
\]

Finally, this gives the answer to our problem:

(5) \[ \int \sec^3 u \, du = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C. \]

6. Integrate (a) \( \int \frac{(x+1) \, dx}{x(x+3)} \) (b) \( \int \frac{(x+1) \, dx}{x^2(x+3)} \)

Answer. For part (a), we seek constants \( A, \ B \) such that

\[ \frac{(x+1)}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}. \]

Put the right hand side over a common denominator and equate numerators, getting \( x + 1 = A(x+3) + Bx \). Now substituting \( x = 0 \), we get \( 1 = 3A \), so \( A = 1/3 \). Substitute \( x = -3 \) to get \(-3 + 1 = B(-3)\), so \( B = 2/3 \). Thus

\[
\int \frac{(x+1) \, dx}{x(x+3)} = \frac{1}{3} \int \frac{dx}{x} + \frac{2}{3} \int \frac{dx}{x+3} = \frac{1}{3} \ln x + \frac{2}{3} \ln(x+3) + C = \frac{1}{3} \ln(x(x+3)^2) + C.
\]

For part (b), we seek constants \( A, \ B, \ C \) such that

\[ \frac{x+1}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}. \]

Put the right hand side over a common denominator and equate numerators, getting \( x + 1 = A(x+3) + B(x+3) + Cx^2 \). Now substituting \( x = 0 \), we get \( 1 = 3B \), so \( B = 1/3 \). Substitute \( x = -3 \) to get \(-3 + 1 = C(9)\), so \( C = -2/9 \). That uses up the roots, so to find \( A \) we have to equate coefficients. Equating the coefficients of \( x^2 \) we get \( 0 = A + C \), so \( A = 2/9 \). Now, we can integrate:

\[
\int \frac{(x+1) \, dx}{x^2(x+3)} = \int \frac{2/9 \, dx}{x} + \int \frac{1/3 \, dx}{x^2} - \int \frac{2/9 \, dx}{x+3} = \frac{2}{9} \ln x - \frac{1}{3} x^{-1} - \frac{2}{9} \ln(x+3) + C.
\]

7. Integrate \( \int \frac{dx}{(x-1)(x+2)^2} \).
Answer. We seek constants $A$, $B$, $C$ such that

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$ 

Put the right hand side over a common denominator and equate numerators, getting $1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$. Substitute $x = 1: 1 = A(9)$, so $A = 1/9$. Substitute $x = -2: 1 = C(-3)$, so $C = -1/3$. Now equate coefficients of $x^2$: $0 = A + B$, so $B = -1/9$. Thus

$$\int \frac{dx}{(x-1)(x+2)^2} = \int \frac{(1/9)dx}{x-1} - \int \frac{(1/9)dx}{x+2} - \int \frac{(1/3)dx}{(x+2)^2} =$$

$$= \frac{1}{9} \ln \frac{x-1}{x+2} + \frac{1}{3} (x+2)^{-1} + C.$$

8. Integrate $\int \frac{(x^2 - 1)dx}{(x^2 + 1)(x+3)}$.

Answer. We seek constants $A$, $B$, $C$ such that

$$\frac{x^2 - 1}{(x^2 + 1)(x+3)} = \frac{A}{x^2 + 1} + \frac{Bx}{x^2 + 1} + \frac{C}{(x+3)}.$$ 

Put the right hand side over a common denominator and equate numerators, getting $x^2 - 1 = A(x+3) + Bx(x+3) + C(x^2 + 1)$. Substitute $x = -3$ to get $(-3)^2 - 1 = C((-3)^2 + 1)$, giving $C = 4/5$. To find $A$ and $B$ we must equate coefficients. For the constant term this gives $-1 = 3A + C$, so $A = -3/5$. Equating coefficients of $x$ gives us $0 = A + 3B$, so $B = -A/3 = 1/5$. Thus

$$\int \frac{(x^2 - 1)dx}{(x^2 + 1)(x+3)} = \int \frac{dx}{x^2 + 1} + \frac{1}{5} \int \frac{xdx}{x^2 + 1} + \frac{4}{5} \int \frac{dx}{x+3}$$

$$= -\frac{3}{5} \arctan x + \frac{1}{10} \ln(x^2 + 1) + \frac{4}{5} \ln(x + 3) + C.$$

9. Integrate $\int \frac{x^2 dx}{\sqrt{9 - x^2}}$.

Answer. We use the substitution indicated in figure 2. Then

$$x = 3 \sin u, \ dx = 3 \cos u \ du, \ \sqrt{9 - x^2} = 3 \cos u.$$
The integral becomes
\[ \int \frac{xdx}{\sqrt{9-x^2}} = \int \frac{(9 \sin^2 u)(3 \cos u)}{3 \cos u} = 9 \int \sin^2 u du. \]
This integral requires the half-angle formula; using that we obtain:
\[ \int \sin^2 u du = \frac{1}{2} \int (1 - \cos(2u))du = \frac{1}{2}(u - \frac{1}{2} \sin(2u)) + C, \]
so
\[ \int \frac{xdx}{\sqrt{9-x^2}} = \frac{9}{2}(u - \frac{1}{2} \sin(2u)) + C. \]
Now, to express this in terms of \( x \), we need the double angle formula \( \sin(2u) = 2 \sin u \cos u \), giving finally:
\[ \int \frac{xdx}{\sqrt{9-x^2}} = \frac{9}{2}(\arcsin \frac{x}{3} - \frac{x\sqrt{9-x^2}}{9}) + C. \]

10. Integrate \( \int \frac{x^2dx}{\sqrt{9+x^2}} \)

Answer. We use the substitution indicated in figure 3.

\[ x = 3 \tan u, \quad dx = 3 \sec^2 u du, \quad \sqrt{9+x^2} = 3 \sec u, \]
and the integral becomes
\[ \int \frac{x^2dx}{\sqrt{9+x^2}} = \int \frac{(9 \tan^2 u)(3 \sec^2 u)}{3 \sec u} = 9 \int \tan^2 u \sec^2 u du. \]
We calculate this integral by parts: \( v = \tan u, \ dv = \sec^2 u du, \ dw = \sec u \tan u du, \ w = \sec u : \)
\[ \int \tan^2 u \sec^2 u du = \tan u \sec u - \int \sec^3 u du \]
\[ = \tan u \sec u - \frac{1}{2} (\sec u \tan u + \ln |\sec u + \tan u|) + C = \frac{1}{2} (\sec u \tan u - \ln |\sec u + \tan u|) + C. \]
Resubstituting back, to get an expression in terms of \( x \):
\[ \int \frac{x^2dx}{\sqrt{9+x^2}} = \frac{9}{2} \left( \frac{x^2}{3} - \ln \left| \frac{\sqrt{9+x^2}}{3} + \frac{x}{3} \right| \right) + C, \quad \text{or} \]
\[ \int \frac{x^2dx}{\sqrt{9+x^2}} = \frac{1}{2} x \sqrt{9+x^2} - \frac{9}{2} \ln |x + \sqrt{9+x^2}| + C. \]