Calculus II
Practice Problems 3: Answers

1. Differentiate:

\[ f(x) = \sqrt{\frac{2x - 6}{3x + 5}}. \]

**Answer.** This problem is here to suggest a different way, called *logarithmic differentiation*, of differentiating expressions like this. First we rewrite the function exponentially:

\[(1) \quad f(x) = (2x - 6)^{1/2}(3x + 5)^{-1/2}\]

and then we take the logarithm:

\[\ln f(x) = \frac{1}{2}[(\ln(2x - 6) - \ln(3x + 5))].\]

Now differentiate this equation:

\[\frac{f'(x)}{f(x)} = \frac{1}{2}\left[\frac{2}{2x - 6} - \frac{3}{3x + 5}\right]\]

and now multiply through by \( f(x) \), as in equation (1):

\[f'(x) = f(x)\left[\frac{1}{2}\left[\frac{2}{2x - 6} - \frac{3}{3x + 5}\right]\right] = (2x - 6)^{-1/2}(3x + 5)^{-1/2} - \frac{3}{2}(2x - 6)^{1/2}(3x + 5)^{-3/2}\]

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2. 

a) \( \tan(\arccos x) = \)

b) \( \frac{1}{x^2} - \tan^2(\arccos x) = \)

**Answer.** a. Draw a right triangle with hypotenuse 1, an angle \( \alpha \), and label the side adjacent to \( \alpha \) as \( x \) so that \( \alpha = \arccos x \) (see the figure).

![Right triangle](image)

Now, by the Pythagorean theorem, the side opposite \( \alpha \) has length \( 1 - x^2 \). Thus

\[\tan(\arccos x) = \tan \alpha = \frac{\sqrt{1 - x^2}}{x}\]
Now, for b):
\[
\frac{1}{x^2} - \tan^2(\arccos x) = \frac{1}{x^2} - \left(\frac{\sqrt{1-x^2}}{x}\right)^2 = \frac{1-(1-x^2)}{x^2} = 1.
\]
From the diagram, \(1/x = \sec \alpha\), so this also follows from the identity \(\tan^2 \alpha + 1 = \sec^2 \alpha\).

3. Differentiate \(y = \arccos \sqrt{x}\)

**Answer.** By the chain rule:
\[
y' = -\frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{x}(1-x)}
\]

4. Differentiate: \(g(x) = \arcsin(\ln x)\).

**Answer.** By the chain rule:
\[
g'(x) = \frac{1}{x\sqrt{1-(\ln x)^2}}
\]

5. Integrate

a) \[
\int_0^2 \frac{xdx}{1+4x^2} = \]

b) \[
\int_0^2 \frac{dx}{1+4x^2} = \]

**Answer.** For a), let \(u = 1+4x^2\), \(du = 8dx\). Then
\[
\int_0^2 \frac{xdx}{1+4x^2} = \frac{1}{8} \int_1^{17} \frac{du}{u} = \frac{1}{8} \ln u \bigg|_1^{17} = \frac{\ln 17}{8}
\]
For b), however, we should make the substitution: \(u = 2x\), \(du = 2dx\). Then
\[
\int_0^2 \frac{dx}{1+4x^2} = \frac{1}{2} \int_0^4 \frac{du}{1+u^2} = \frac{1}{2} \arctan u \bigg|_0^4 = \frac{\arctan 4}{2}
\]
This problem illustrates that one must be careful in deciding what substitution to make. One has to scan the integrand to see how to write it as a product of a function of \(u\) (the substitution to be made) and the differential of \(u\).

6. Integrate:

a) \[
\int \frac{e^x dx}{e^{2x}+1} = \]

b) \[
\int \frac{dx}{e^x + e^{-x}} = \]

**Answer.** For part a) recognize that for \(u = e^x\), \(du = e^x dx\), and thus
\[
\int \frac{e^x dx}{e^{2x}+1} = \int \frac{du}{1+u^2} = \arctan u + C = \arctan e^x + C.
\]
For part b), we see that, in order to make the same substitution, we need an \( e^x \) in the numerator. So put it there, by multiplying by \( e^x / e^x = 1 \):

\[
\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{e^x e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x} + 1} = \arctan e^x + C,
\]

by part a).

7. \( \int \frac{dx}{\sqrt{5 - 4x - x^2}} = \)

**Answer.** To get this to look like something recognizable, we complete the square: \( 5 - 4x - x^2 = 9 - (x + 2)^2 \). The problem now looks like

\[
\int \frac{dx}{\sqrt{9 - (x + 2)^2}} = .
\]

This looks like the integral should involve \( \arccos \), but we need a 1 where the 9 is. We fix that by the substitution \( 3u = x + 2 \), \( 3du = dx \). Then we have

\[
\int \frac{dx}{\sqrt{9 - (x + 2)^2}} = \int \frac{3du}{\sqrt{9 - 9u^2}} = \int \frac{du}{\sqrt{1 - u^2}} = -\arccos u + C = -\arccos \frac{x + 2}{3} + C.
\]

8. Integrate:

a) \( \int \tan^2 x dx = \)

**Answer.** \( \tan^2 x = \sec^2 x - 1 \), so

\[
\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C.
\]

b) \( \int \tan^3 x dx = \)

**Answer.** Using the same identity

\[
\int \tan^3 x dx = \int (\sec^2 x - 1) \tan x dx = \int \sec x (\sec x \tan x) dx - \int \tan x dx
\]

\[
= \frac{\sec^2 x}{2} + \ln(\cos x) + C,
\]

using the substitution \( u = \sec x \), \( du = \sec x \tan x dx \) for the first integral.

9. \( \int x \tan(x^2 + 1) dx = \)

**Answer.** Let \( u = x^2 + 1 \), \( du = 2xdx \). Then

\[
\int x \tan(x^2 + 1) dx = \frac{1}{2} \int \tan u du = -\frac{1}{2} \ln(\cos u) + C = -\frac{1}{2} \ln(\cos(x^2 + 1)) + C.
\]
10. \[ \int \frac{dx}{x^2 - 6x + 13} = \]

By completing the square, we see that \( x^2 - 6x + 13 = (x - 3)^2 + 4 \). Let \( u = x - 3 \), \( du = dx \), so that we have

\[ \int \frac{dx}{x^2 - 6x + 13} = \int \frac{dx}{(x - 3)^2 + 4} = \int \frac{du}{u^2 + 4}. \]

Now, we know the integral of \( du/(u^2 + 1) \); but how do we do \( du/(u^2 + 4) \)? Again a little algebra comes into play:

\[ u^2 + 4 = 4((\frac{u}{2})^2 + 1). \]

Try the substitution \( v = u/2 \), \( dv = du/2 \):

\[ \int \frac{du}{u^2 + 4} = \frac{1}{4} \int \frac{2dv}{v^2 + 1} = \frac{1}{2} \arctan v + C = \frac{1}{2} \arctan \frac{x - 3}{2} + C. \]