Calculus II
Practice Problems 13: Answers

1. A man drops out of a plane at 25,000 feet of altitude and immediately opens his parachute. For this man and parachute the deceleration due to air resistance is $4v$ where v is his velocity. How far has he fallen in one minute? How long does it take for him to hit the ground, and at what velocity does he hit the ground?

Answer. Let $x(t)$ be the distance the man has fallen in time t, and $x'(t)$ his velocity. The initial conditions are $x(0) = 0,\ x'(0) = 0$. Now, his acceleration is due to gravity (32 ft/sec^2) less that due to air resistance, giving the differential equation

$$x'' = 32 - 4x' \qquad (2)$$

The homogeneous equation ($x'' + 4x' = 0$) has the general solution $x_h = A + Be^{-4t}$. To find a particular solution of (2), we try$x_p = at + b$. We get $0 = 32 - 4a$, so $a = 8$ and our solution is

$$x(t) = 8t + A + Be^{-4t}$$

The initial conditions give $0 = A + B,\ 0 = 8 - 4B$, so $B = 2,\ A = -2$ and the equation of motion is

$$x(t) = 8t - 2 + 2e^{-4t}$$

Now, we want to find the value of t when $x(t) = 25,000$. At this distance, the terms $-2 + 4e^{-4t}$ are negligible, so the answer is $t = 25,000/8 = 52$ seconds, give or take at most a tenth of a second. The point

of this problem is that, except for the first second or so, the man is falling at (essentially) a constant velocity of 8 ft/sec. A more interesting, and realistic, problem would be to have the man free-fall for about 5 seconds, and then pull the parachute. After 5 seconds, with the acceleration due only to gravity, the man's velocity is 160 ft/sec. When the parachute opens, he then decelerates abruptly to (nearly) the velocity of 8 ft/sec.

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2. a) Let a mass m hang from a spring of spring constant k. Suppose that it is set in motion. Show that, throughout the motion, $mv^2 + kx^2$ is constant, where x represents displacement from equilibrium, and v is velocity.

Answer. The equation of motion is $mx'' + kx = 0$. Multiply by dx/dt, noting that $v = dx/dt$ and $dv/dt = x''$. We get

$$mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0$$

which integrates to $mv^2 + kx^2 = C$.

b) Suppose that $k = 4$ dynes/cm and $m = 10$ g, and the spring is already in motion At a particular instant the srping is located 10 cm. from equilibrium and traveling at velocity 8 cm/sec. For this motion, what are the maximum velocity and maximum displacement of the mass?

Answer. According to part a) we have that $10v^2 + 4x^2$ is constant. At the instant given, $x = 10$ and $v = 8$, so the constant is $10(8)^2 + 4(10)^2 = 1040$. Now the velocity is a maximum when $x = 0$, so we have $10(v_{max})^2 = 1040$, so that $v_{max} = 10.2$ cm/sec. Similarly, displacement is a maximum when $v = 0$, so $4(x_{max})^2 = 1040$, and $x_{max} = 16.12$ cm.

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3. The above configuration is put in a viscous fluid which exerts a retardant force proportional to the velocity, with constant of proportionality $q = 12$. Find the equation of motion of the mass, given that at time $t = 0$ it is

at $x=0$ and its velocity is 4.8 cm/sec. What is the maximum displacement of the mass?

Answer. The differential equation of motion is

$$10x'' + 12x' + 4x = 0$$

The roots of the auxiliary equation are $-.6 \pm .2i$, so the general solution is

$$x = e^{-.6t}(A\cos(.2t) + B\sin(.2t))$$

The initial conditions give us $A = 0,\ .2B = 4.8$, so $B = 24$, and the equation of motion is

$$x = 24e^{-1.2t}\sin(.2t)$$

To find the maximum value of x we find the first value of t for which $dx/dt = 0$. Now

$$\frac{dx}{dt} = 24(-1.2e^{-1.2t}\sin(.2t) + .2e^{-1.2t}\cos(.2t))$$

This is zero when $\tan(.2t) = .2/1.2$, giving $t = .8257$ sec. Then the maximum displacement is

$$x = 24e^{-1.2(.8257)}\sin(.2(.8257)) = 1.4645 \text{ cm.}$$

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4. A crystal glass consists of cells in a crystalline shape which oscillate at a natural frequency, so the motion is governed by a differential equation $x'' + \omega_0^2 x = 0$ where $2\pi/\omega_0$ is the frequency. If the ambient air is vibrating at a frequency of $\omega/2\pi$ (due to a monotonal sound, perhaps), then the motion of a crystal is modified by the force of the air in motion so as to be governed by the inhomogeneous equation

$$x'' + \omega_0^2 x = A\cos\omega t$$

Find a particular solution of this differential equation. What happens as ω approaches ω_0? (This phenomenon is called *resonance*.)

Answer. We try $y = a\cos\omega t + b\sin\omega t$. Substituting that in the differential equation leads to

$$-\omega^2(a\cos\omega t + b\sin\omega t) + \omega_0^2(a\cos\omega t + b\sin\omega t) = A\cos\omega t$$

leading to the values

$$a = \frac{A}{\omega_0^2 - \omega^2},\ \ b = 0,$$

so the particular solution is

$$x_p = \frac{A}{\omega_0^2 - \omega^2}\cos\omega t\ .$$

As we can see, as $\omega \to \omega_0$, the amplitude of this wave becomes arbitrarily large. This is why a singer can crack a crystal glass by singing a note at a frequency very close to the natural frequency of the crystal.

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5. Consider the circuit as shown in the diagram. The switch is turned on at time $t = 0$. Find the current as a function of time.

Answer. The circuit equation is $LI'' + RI' + (1/C)I = E'(t)$. In our case we have $L = .2,\ R = 10,\ 1/C = 500,\ E'(t) = 60\cos(60t)$, so the equation is

$$\frac{1}{5}I'' + 10I' + 500I = 60\cos(60t)\ .$$

To find a particular solution we try

$$I = A\cos(60t) + B\sin(60t)\ .$$

We calculate the derivatives:

$$I' = -60A\sin(60t) + B\cos(60t)I\ , \quad I'' = -3600A\cos(60t) + 3600B\sin(60t)\ ,$$

and substitute in the differential equation to find A and B. We get the equations

$$-720A + 600B + 500A = 60\ , \quad -720B - 600A + 500B = 0\ ,$$

which have the solutions $A = -.323,\ B = .088$. So

$$I_p(t) = -.323\cos(60t) + .088\sin(60t)$$

as a particular solution. Now to find the solution for our initial value problem, we look at the homogeneous equation. The roots are $-25 \pm 25i\sqrt{3}$, so we try to fit

$$I(t) = e^{-25t}(a\cos(25\sqrt{3}t) + b\sin(25\sqrt{3}t)) - .323\cos(60t) + .088\sin(60t)$$

to the initial values. At the time the switch is turned on, $I(0) = 0,\ I'(0) = 0$. We find $a = .032$ and $b = .103$, so the answer is

$$I(t) = e^{-25t}(.032\cos(25\sqrt{3}t) + .103\sin(25\sqrt{3}t)) - .323\cos(60t) + .088\sin(60t)\ .$$

Because of the exponent e^{-25t}, the first term becomes negligible after a while, and the current oscillates with frequency $60/2\pi$ and amplitude $\sqrt{(.323)^2 + (.088)^2} = .334$ amperes.