Calculus II
Practice Problems 12: Answers

1. Solve $y'' - 5y = 0$ with the initial values $y(0) = 1,\ y'(0) = -1$.

Answer. The auxiliary equation $r^2 - 5 = 0$ has the roots $r = \pm\sqrt{5}$, so the general solution is

$$y = A\cosh\sqrt{5}x + B\sinh\sqrt{5}x\ .$$

The initial conditions give us $A = 1,\ \sqrt{5}B = -1$. Thus the answer is

$$y = \cosh\sqrt{5}x - \frac{1}{\sqrt{5}}\sinh\sqrt{5}x\ .$$

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2. Solve $y'' + 5y = 0$ with the initial values $y(0) = 1,\ y'(0) = -1$.

Answer. The auxiliary equation $r^2 + 5 = 0$ has the roots $r = \pm i\sqrt{5}$, so the general solution is

$$y = A\cos\sqrt{5}x + B\sin\sqrt{5}x\ .$$

The initial conditions give us $A = 1,\ \sqrt{5}B = -1$. Thus the answer is

$$y = \cos\sqrt{5}x - \frac{1}{\sqrt{5}}\sin\sqrt{5}x\ .$$

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3. Solve $y'' - 5y' + 6y = 0$ with the initial values $y(0) = 1,\ y'(0) = -1$.

Answer. The auxiliary equation $r^2 - 5r + 6 = 0$ has the roots $r = 2,\ 3$, so the general solution is

$$y = Ae^{2x} + Be^{3x}\ .$$

The initial conditions give the equations

$$1 = A + B\ , \qquad -1 = 2A + 3B\ , \quad \text{so that} \quad A = 4\ , \quad B = -3$$

and the solution is

$$y = 4e^{2x} - 3e^{3x}\ .$$

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4. Solve $y'' + 4y' + 5y = 0$ with the initial values $y(0) = 1,\ y'(0) = -1$.

Answer. The auxiliary equation $r^2 + 4r + 5 = 0$ has the roots $r = -2 \pm i$, so the general solution is

$$y = e^{-2x}(A\cos x + B\sin x)$$

with derivative

$$y' = -2e^{-2x}(A\cos x + B\sin x) + e^{-2x}(-A\sin x + B\cos x).$$

The initial conditions give the equations $1 = A,\ -1 = -2A + B$, or $A = 1, B = 1$, so the solution is $y = e^{-2x}(\cos x + \sin x)$.

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5. Solve $y'' - y' = 0$ with the initial values $y(2) = 1,\ y'(2) = 2$.

Answer. The auxiliary equation $r^2 - r = 0$ has the roots $r = 0,\ 1$, so the general solution is $y = A + Be^x$. The initial conditions give the equations $1 = A + Be^2,\ 2 = Be^2$, so $B = 2/e^2,\ A = -1$, and the solution is $y = -1 + 2e^{x-2}$.

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6. Solve $y'' + 2y' + y = 0$ with the values $y(-1) = 1,\ y(1) = 1$.

Answer. The auxiliary equation $r^2 + 2r + 1 = 0$ has the single root $r = -1$, so the general solution is $y = Ae^{-x} + Bxe^{-x}$. The specified values (called *boundary values*) give us the equations:

$$1 = Ae - Be\ , \quad 1 = Ae^{-1} + Be^{-1}$$

with solutions $A = (e + e^{-1})/2,\ B = (e - e^{-1})/2$, and the desired function is

$$y = e^{-x}(\frac{e + \frac{1}{e}}{2} + \frac{e - \frac{1}{e}}{2}x)$$

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7. Solve $y'' + 2y' + y = x$ with the initial values $y(0) = 0,\ y'(0) = 0$.

Answer. First we find the general solution of the homogeneous equation. Its auxiliary equation is $r^2 + 2r + 1$, which has the single root $r = -1$. Thus we have

(1) $$y_h = e^{-x}(Ax + B)\ .$$

Now, we find a particular solution by trying $y = ax + b$. Substituting that into the given equation gives us $0 + 2a + ax + b = x$, so we must have $a = 1,\ 2a + b = 0$ so $b = -2$, and $y_p = x - 2$. Thus our solution is of the form

$$y = y_p + y_h = x - 2 + e^{-x}(Ax + B) \quad \text{with} \quad y' = 1 - e^{-x}(Ax + B) + Ae^{-x}\ .$$

We solve for A and B using the initial conditions;

$$0 = -2 + B, 0 = 1 - B + A \quad \text{or} \quad B = 2,\ A = 1\ .$$

The solution is

$$y = x - 2 + e^{-x}(x + 2)$$

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8. Find the general solution of $y'' + 2y' + y = \sin x$.

Answer. We have the same solutions (1) of the homogeneous equation as above. To find a particular solution, we try $y = a\cos x + b\sin x$, leading to the equation:

$$-a\cos x - b\sin x + 2(-a\sin x + b\cos x) + a\cos x + b\sin x = \sin x\ .$$

Equating coefficients: $-a + 2b + a = 0,\ -b - 2a + b = 1$, so $a = -1/2,\ b = 0$. Thus the general solution is

$$y = y_p + y_h = -\frac{1}{2}\cos x + e^{-x}(Ax + B)\ .$$

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9. Find the general solution of $y'' - 4y = \sin(2x)$.

Answer. The auxiliary equation for the homogeneous equation is $r^2 - 4 = 0$, which has the roots $r = \pm 2$. Thus the general solution of the homogeneous equation is $Ae^{2x} + Be^{-2x}$. To find a particular solution of our equation we try $y_p = a\sin(2x) + b\cos(2x)$. Substituting this in the given equation leads to

$$-4a\sin(2x) - 4b\cos(2x) - 4(a\sin(2x) + b\cos(2x)) = \sin(2x)$$

giving us $y_p = -(1/8)\sin(2x)$. Thus the answer is

$$y = -\frac{1}{8}\sin(2x) + Ae^{2x} + Be^{-2x}$$

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10. Find the general solution of $y'' + 4y = \sin(2x)$.

Answer. Here the inhomogeneous term satisfies the homogeneous equation, so we have to try a function of the form

$$y = (ax+b)\sin(2x) + (cx+d)\cos(2x)$$

We find

$$y'' = -4(ax+b)\sin(2x) + 4a\cos(2x) - 4(cx+d)\cos(2x) - 4C\sin(2x)$$

and thus

$$y'' + 4y = 4a\cos(2x) - 4c\sin(2x) = \sin(2x)$$

Thus $a =,\ c = -1/4$ and b and d can be anything. The general solution is

$$y = b\sin(2x) + (-\frac{1}{4}x + d)\cos(2x)$$