1. Since \( \tan(\pi/6) = 1/\sqrt{3} \), and therefore, \( \pi = 6 \arctan(1/\sqrt{3}) \), we can use the Taylor series for the arctangent to estimate \( \pi \). Do this, using the first three nonzero terms.

**Answer.** We have the Taylor expansion:

\[
\arctan \frac{1}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2n+1},
\]

so the first three nonzero terms are

\[
\frac{1}{\sqrt{3}} - \frac{1}{3} \left( \frac{1}{\sqrt{3}} \right)^3 + \frac{1}{5} \left( \frac{1}{\sqrt{3}} \right)^5 = \frac{1}{\sqrt{3}}(1 - \frac{1}{27} + \frac{1}{45}) = .568797,
\]

Since this estimates \( \pi/6 \), the estimate for \( \pi \) is 3.41278.

---

2. Since \( \sin(\pi/6) = 1/2 \), we can also find \( \pi \) by solving the equation \( \sin x = 1/2 \). We can approximate the solution by replacing \( \sin \) by an approximating Taylor polynomial, and then using Newton’s method. Do this with the three term Taylor polynomial for \( \sin x \).

**Answer.** The three term Taylor polynomial for \( \sin x \) is

\[
f(x) = x - \frac{x^3}{6} + \frac{x^5}{120}.
\]

We want to find the value of \( x \) for which this is 1/2. We thus apply Newton’s method for \( f(x) - 1/2 \). The recursion formula is

\[
x' = x - \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{1}{2}}{1 - \frac{x^2}{2} + \frac{x^4}{24}}
\]

Starting with the guess \( x_0 = .5 \), we obtain recursively

\[
x_1 = 0.523442136, \quad x_2 = 0.523596306, \quad x_3 = 0.523596313
\]

so we have stability after three steps. Since this approximates \( \pi/6 \), the estimate for \( \pi \) that we get is 3.141577879.

---

3. Find a solution, by Newton’s method, of the equation

\[
x^5 - x^4 + x^3 - x^2 - 4 = 0
\]

correct to five decimal places.

**Answer.** Here the recursion is

\[
x' = x - \frac{x^5 - x^4 + x^3 - x^2 - 4}{5x^4 - 4x^3 + 3x^2 - 2x}
\]

Starting with \( x_0 = 1 \), we obtain the sequence

\[
x_1 = 3, \quad x_2 = 2.446540881, \quad x_3 = 2.023831867, \quad x_4 = 1.729251795
\]

\[
x_5 = 1.570080738, \quad x_6 = 1.524684789, \quad x_7 = 1.521396, \quad x_8 = 1.521379707.
\]
The next value repeats the last value, so this is the desired approximation. You will see that if you start with practically any initial value \((x_0 = 0\) won’t do - why?) you end up with the same root. Does that tell us that there is only one real root to this equation?

4. Here is another way of estimating \(\pi\). We know that

\[
\pi/4 = \int_0^1 \frac{dx}{1 + x^2}.
\]

Estimate this integral by the trapezoid rule, using steps of size 1/10. How many steps should we take to be sure of an estimate correct to 4 decimal places?

**Answer.** We have to evaluate \(1/(1 + x^2)\) at each of the points 0, 0.1, 0.2, 0.3, ..., 1. The trapezoid rule then gives us as the estimate

\[
\frac{1}{20}(1 + 2(0.9900 + 0.96153 + 0.91743 + 0.86206 + 0.8 + 0.73529 + 0.6711 + 0.60975 + 0.55248) + .5)
\]

which is 3.1392. To be correct to within 4 decimal places, we need \(N\) to satisfy

\[
E(N) = \frac{(1 - 0)^3}{12N^2} M_2 < \frac{1}{2} 10^{-4}.
\]

where \(M_2\) bounds the second derivative. We calculate that we can take \(N\) so that \(6N^2 > 10^4\). \(N = 50\) will do. Note that I just need a big enough \(N\), not the smallest which will do: the computer can do this calculation for any \(N < 1000\) in just about the same time.

5. Define

\[
J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}.
\]

Evaluate \(J_0(1)\) correctly to 4 decimal places.

**Answer.** Although this looks like something made up by a mad mathematician, it happens to be the first in a sequence of functions important in engineering, known as the Bessel functions. Let’s write down the first few terms of \(J_0(x)\):

\[
J_0(x) = 1 - \frac{x^2}{8} + \frac{x^4}{16 \cdot 2 \cdot 6} - \frac{x^6}{64 \cdot 6 \cdot 24} + \cdots
\]

Now, we have the problem of finding out how many terms to take to get a 10^{-4} estimate. We can’t turn to Taylor’s error estimate, for that requires estimates on the successive derivatives of \(J_0\), and since we’ll only have a series expression for these, the problem is only magnified. But, we observe that the series for \(J_0(1)\) is an alternating series with decreasing general term, so the error between any partial sum and the true value is less than the next term. The next term of this series is

\[
\frac{1}{4^4 (4!)^2} = (737280)^{-1} < 10^{-5},
\]

so we get our estimate by evaluating the four terms above:

\[
J_0(1) = 1 - \frac{1}{8} + \frac{1}{16 \cdot 2 \cdot 6} - \frac{1}{64 \cdot 6 \cdot 24} = .8801
\]
6. Find an estimate for
\[ \int_0^2 \frac{\sin x}{x} \, dx \]
using Simpson’s rule with \( N = 20 \) subdivisions.

**Answer.** We divide the interval \([0, 2]\) into tenths, and evaluate \(\frac{\sin x}{x}\) at all of the endpoints. Of course, the attempt to evaluate this at \(x = 0\) fails, but we can take that value to be 1 since
\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]

Simpson’s rule gives us \((\sin x)/x = 0.80270\) correct to four decimals.