Calculus II
Problems on Numerical Methods, Answers

1. Since $\tan(\pi/6) = 1/\sqrt{3}$,and therefore, $\pi = 6\arctan(1/\sqrt{3})$ we can use the Taylor series for the arc tangent to estimate π. Do this, using the first three nonzero terms.

Answer. We have the Taylor expansion:

$$\arctan\frac{1}{\sqrt{3}} = \sum_{n=0}^{\infty}(-1)^n \frac{\frac{1}{\sqrt{3}}^{2n+1}}{2n+1} ,$$

so the first three nonzero terms are

$$\frac{1}{\sqrt{3}} - \frac{1}{3}(\frac{1}{\sqrt{3}})^3 + \frac{1}{5}(\frac{1}{\sqrt{3}})^5 = \frac{1}{\sqrt{3}}(1 - \frac{1}{27} + \frac{1}{45}) = .568797 ,$$

Since this estimates $\pi/6$, the estimate for π is 3.41278.

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2. Since $\sin(\pi/6) = 1/2$, we can also find π by solving the equation $\sin x = 1/2$. We can approximate the solution by replacing sin by an approximating Taylor polynomial, and then using Newton's method. Do this with the three termTaylor polynomial for $\sin x$.

Answer. The three term Taylor polynomial for $\sin x$ is

$$f(x) = x - \frac{x^3}{6} + \frac{x^5}{120} .$$

We want to find the value of x for which this is $1/2$. We thus apply Newton's method for $f(x) - 1/2$. The recursion formula is

$$x' = x - \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{1}{2}}{1 - \frac{x^2}{2} + \frac{x^4}{24}}$$

Starting with the guess $x_0 = .5$, we obtain recursively

$$x_1 = 0.523442136 , \quad x_2 = 0.523596306 , \quad x_3 = 0.523596313$$

so we have stability after three steps. Since this approximates $\pi/6$, the estimate for π that we get is 3.141577879.

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3. Find a solution, by Newton's method, of the equation

$$x^5 - x^4 + x^3 - x^2 - 4 = 0$$

correct to five decimal places.

Answer. Here the recursion is

$$x' = x - \frac{x^5 - x^4 + x^3 - x^2 - 4}{5x^4 - 4x^3 + 3x^2 - 2x}$$

Starting with $x_0 = 1$, we obtain the sequence

$$x_1 = 3 , \quad x_2 = 2.446540881 , \quad x_3 = 2.023831867 , \quad x_4 = 1.729251795$$

$$x_5 = 1.570080738 , \quad x_6 = 1.524684789 , \quad x_7 = 1.521396 , \quad x_8 = 1.521379707 .$$

The next value repeats the last value, so this is the desired approximation. You will see that if you start with practically any initial value ($x_0 = 0$ won't do - why?) you end up with the same root. Does that tell us that there is only one real root to this equation?

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4. Here is another way of estimating π. We know that

$$\pi/4 = \int_0^1 \frac{dx}{1+x^2} \ .$$

Estimate this integral by the trapezoid rule, using steps of size $1/10$. How many steps should we take to be sure of an estimate correct to 4 decimal places?

Answer. We have to evaluate $1/(1+x^2)$ at each of the points $0, 0.1, 0.2, 0.3, \ldots, 1$. The trapezoid rule then gives us as the estimate

$$\frac{1}{20}(1+2(0.9900+0.96153+0.91743+0.86206+0.8+0.73529+0.6711+0.60975+0.55248)+.5)$$

which is 3.13992.To be correct to within 4 decimal places, we need $N$to satisfy (using the error formula; see proposition 10.5)

$$E(N) = \frac{(1-0)^3}{12N^2}M_2 < \frac{1}{2}10^{-4} \ .$$

where M_2 bounds the second derivative. We calculate that we can take N so that $6N^2 > 10^4$. $N = 50$ will do. Note that I just need a big enough N, not the smallest which will do: the computer can do this calculation for any $N < 1000$ in just about the same time.

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5. Define

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)(n+1)!} \ .$$

Evaluate $J_0(1)$ correctly to 4 decimal places.

Answer. Although this looks like something made up by a mad mathematician, it happens to be the first in a sequence of functions important in engineering, known as the *Bessel* functions. Let's write down the first few terms of $J_0(x)$:

$$J_0(x) = 1 - \frac{x^2}{8} + \frac{x^4}{16\cdot 2\cdot 6} - \frac{x^6}{64\cdot 6\cdot 24} + \cdots$$

Now, we have the problem of finding out how many terms to take to get a 10^{-4} estimate. We can't turn to Taylor's error estimate, for that requires estimates on the successive derivatives of J_0, and since we'll only have a series expression for these, the problem is only magnified. But, we observe that the series for $J_0(1)$ is an alternating series with decreasing general term, so the error between any partial sum and the true value is less than the next term. The next term of this series is

$$\frac{1}{4^4(4!)(5!)} = (737280)^{-1} < 10^{-5} \ ,$$

so we get our estimate by evaluating the four terms above:

$$J_0(1) = 1 - \frac{1}{8} + \frac{1}{16\cdot 2\cdot 6} - \frac{1}{64\cdot 6\cdot 24} = .8801$$

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6. Find an estimate for

$$\int_0^2 \frac{\sin x}{x} dx$$

using Simpson' s rule with $N = 20$ subdivisions.

Answer. We divide the interval $[0,2]$ into tenths, and evaluate $\sin x/x$ at all of the endpoints. Of course, the attempt to evaluate this at $x = 0$ fails, but we can take that value to be 1 since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \ .$$

Simpson's rule gives us $(\sin x)/x = 0.80270$ correct to four decimals.