Calculus II
Practice Exam 4, Answers

1. Find the center, vertices and foci of the ellipse given by the equation $x^2 + 12y^2 - 6x = 15$.

Answer. Complete the square:

$$(x-3)^2 + 12y^2 = 24$$

giving the standard form

$$\frac{(x-3)^2}{24} + \frac{y^2}{2} = 1 \ .$$

The center is at $(3,0)$, and the axis is the line $y = 0$. Since $a^2 = 24$, the vertices are at $(3 \pm \sqrt{24}, 0)$. Now $c^2 = a^2 - b^2 = 24 - 2 = 22$, so the foci are at $(3 \pm \sqrt{22}, 0)$.

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2. Consider the parabola $y^2 = 16(x+1)$.

a) What are the coordinates of the vertex V and the focus F?

Answer. The vertex is at $(-1,0)$. Since $4p = 16$, and the parabola opens to the right, the focus is 4 units to the right of the vertex, at $(3,0)$.

b) Find a point P on the parabola at which the tangent line makes an angle of $45°$ with the line joining P to F.

Answer. By the optical property of the parabola, the tangent at P makes an angle of $45°$ with the horizontal, so has slope $\tan 45° = 1$. Differentiating the equation of the curve:

$$2y\,dy = 16dx \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{8}{y} \ .$$

Thus $dy/dx = 1$ when $y = 8$ and $x = 3$. The point is thus $(3,8)$.

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3. Find the equation of the hyperbola with vertices at $(-1,-3)$ and $(-1,5)$ and foci at $(-1,-4)$ amd $(-1,6)$.

Answer. Since the vertices are on the line $x = -1$, that is the axis of the hyperbola. The center is midway between the vertices, at $(-1,1)$, and $b = 4$, $c = 5$. Thus $a^2 = c^2 - b^2 = 9$, and the equation is

$$-\frac{(x+1)^2}{9} + \frac{(y-1)^2}{16} = 1 \ .$$

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4. Find an integral (do not try to evaluate it) giving the length of the curve $r = \sqrt{\cos(\theta)}$ from $\theta = -\pi/2$ to $\theta = \pi/2$.

Answer. Differentiating, we have

$$\frac{dr}{d\theta} = \frac{-\sin\theta}{2\sqrt{\cos\theta}} \ ,$$

so

$$\frac{ds}{d\theta} = \sqrt{r^2 + (r')^2} = \sqrt{\cos\theta + \frac{\sin^2\theta}{4\cos\theta}} = \sqrt{\frac{1 + 3\cos^2\theta}{4\cos\theta}}$$

and thus

$$Length = \int_{-\pi/2}^{\pi/2} \sqrt{\frac{1+3\cos^2\theta}{4\cos\theta}}\, d\theta$$

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5. Find the area swept out by the line segment $r = 1/\theta$ as θ ranges from 2π to 4π.

Answer.

$$Area = \frac{1}{2}\int_{2\pi}^{4\pi} r^2 d\theta = \frac{1}{2}\int_{2\pi}^{4\pi} \theta^{-2} d\theta = -\frac{1}{2}\theta^{-1}\Big|_{2\pi}^{4\pi} = \frac{1}{8\pi}$$