Calculus I
Final Exam, Spring 2003, Answers

1. Find the integrals:

a) $\int (e^{\sin x})^2 \cos x dx$

Answer. Let $u = \sin x,\ du = \cos x dx$. Then

$$\int (e^{\sin x})^2 \cos x dx = \int (e^u)^2 du = \int e^{2u} du = \frac{e^{2u}}{2} + C = \frac{e^{2\sin x}}{2} + C\ .$$

Alternatively, let $v = e^{\sin x},\ dv = e^{\sin x} \cos x dx$, so that

$$\int (e^{\sin x})^2 \cos x dx = \int e^{\sin x}(e^{\sin x} \cos x dx) = \int v dv = \frac{v^2}{2} + C = \frac{(e^{\sin x})^2}{2} + C\ .$$

b) $\int x\sqrt{x-1} dx$

Answer. Let $u = x - 1$ so that $x = u + 1$ and $du = dx$. Then

$$\int x\sqrt{x-1} dx = \int (u+1)u^{1/2} du = \int (u^{3/2} + u^{1/2}) du = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C\ .$$

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2. Integrate $\int \frac{t^2}{(t^2-1)(t-2)} dt$

Answer. . We expand the function in partial fractions. The roots are -1,1,2, so we write

$$\frac{t^2}{(t^2-1)(t-2)} = \frac{A}{t+1} + \frac{B}{t-1} + \frac{C}{t-2} = \frac{A(t-1)(t-2) + B(t+1)(t-2) + C(t^2-1)}{(t^2-1)(t-2)}\ .$$

Equate the numerators at the roots.

$$t = -1: \quad (-1)^2 = A(-2)(-3) \quad \text{so} \quad A = \frac{1}{6}$$

$$t = 1: \quad 1^2 = B(2)(-1) \quad \text{so} \quad B = -\frac{1}{2}$$

$$t = 2: \quad 2^2 = C(4-1) \quad \text{so} \quad C = \frac{4}{3}$$

This gives us

$$\int \frac{t^2}{(t^2-1)(t-2)} dt = \frac{1}{6}\int \frac{dt}{t+1} - \frac{1}{2}\int \frac{dt}{t-1} + \frac{4}{3}\int \frac{dt}{t-2}$$
$$= \frac{1}{6}\ln(t+1) - \frac{1}{2}\ln(t-1) + \frac{4}{3}\ln(t-2) + C\ .$$

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3. Integrate $\int x \ln x dx$

Answer. We integrate by parts so as to get rid of the ln term: $u = \ln x,\ dv = x dx$, so that $du = dx/x,\ v = x^2/2$, and

$$\int x \ln x dx = \frac{x^2}{2}\ln x - \int (\frac{x^2}{2})\frac{dx}{x} = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C\ .$$

4. A certain compound transforms from state A to state B at a (per minute) rate proportional to the concentration of B in the mixture:

$$\frac{dc_A}{dt} = -.02c_B \ ,$$

where c_A and c_B are the concentrations of A and B respectively (and, assuming no other material is present, $c_A + c_B = 1$). If at time $t = 0$ the mixture is 90% in state A how long will it take to be 10% A?

Answer. Substitute $c_B = 1 - c_A$ in the differential equation, and separate variables, obtaining

$$\frac{dc_A}{1 - c_A} = -.02dt \ .$$

Integrate both sides and exponentiate:

$$-\ln(1 - c_A) = -.02t + C \quad \text{exponentiating to} \quad 1 - c_A = Ke^{.02t} \ .$$

Solve for K using $c_A = .9$ when $t = 0$, getting $K = .1$. Now solve for c_A:

$$c_A = 1 - .1e^{.02t} \ .$$

(Of course this makes sense so long as $c_A > 0$; once A is gone, the process stops.) Now set $c_A = .1$ and solve for t: $.1e^{.02t} = .9$, so

$$t = \frac{\ln 9}{.02} = 109.86 \ \text{minutes} \ .$$

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5. Find the limit. Show your work.

a) $\lim_{x \to 1} \frac{\ln x}{\sin(\pi x)} =$

Answer. At $x = 1$, both numerator and denominator are zero, so l'Hôpital's rule applies:

$$\lim_{x \to 1} \frac{\ln x}{\sin(\pi x)} =^{l'H} \lim_{x \to 1} \frac{1/x}{\pi \cos(\pi x)} = -\frac{1}{\pi} \ .$$

b) $\lim_{x \to 0} \frac{xe^x}{e^{2x} - 1} =$

Answer. Again both numerator and denominator are zero at $x = 0$, so

$$\lim_{x \to 0} \frac{xe^x}{e^{2x} - 1} =^{l'H} \lim_{x \to 0} \frac{e^x + xe^x}{2e^{2x}} = \frac{1}{2} \ .$$

c) $\lim_{x \to \infty} \frac{3x^6 + 7x^4}{2(x^3 + 1)^2} = \frac{3}{2}$

since the factors have the same degree.

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6. Do the integrals converge? If so, evaluate:

a) **Answer**. $\int_0^\infty xe^{-x}dx = \lim_{A \to \infty} \int_0^A xe^{-x}dx = \lim_{A \to \infty} (xe^{-x} - e^{-x})\Big|_0^A = \lim_{A \to \infty} (e^{-A}(A - 1) - (-1)) = 1$.

b) $\int_2^\infty \frac{dx}{x(\ln x)^{25}}$

Answer. Let $u = \ln x,\ du = dx/x$. Then

$$\int_2^A \frac{dt}{x(\ln x)^{25}} = \int_{\ln 2}^A \frac{du}{u^{25}} = -\frac{u^{-24}}{24}\Big|_{\ln 2}^A = \frac{1}{24}\left(\frac{1}{(\ln 2)^{24}} - \frac{1}{(\ln A)^{24}}\right)$$

which converges to $1/(24(\ln 2)^{24})$ as $A \to \infty$.

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7. Do the series converge or diverge? Give a valid reason for your answer.

a) $\sum_{n=1}^{\infty} \frac{n^2+1}{(n+1)^3}$

Answer. The series diverges by comparison with the p-test with $p = 1$: the denominator is only of degree 1 more than the numerator.

b) $\sum_{n=1}^{\infty} \frac{\ln n}{2^n}$

Answer. The series converges by comparison with a geometric series:

$$\frac{\ln n}{2^n} \le \frac{2^{n/2}}{2^n} = \left(\frac{1}{\sqrt{2}}\right)^n ,$$

and $1/\sqrt{2} < 1$.

c) $\sum_{n=1}^{\infty} \frac{(n!+1)^2}{((n+1)!)^2}$

Answer. The series converges by comparison with the p-test with $p = 2$. Divide both numerator and denominator by $(n!)^2$:

$$\frac{(n!+1)^2}{((n+1)!)^2} = \frac{(1+\frac{1}{n!})^2}{(\frac{(n+1)!}{n!})^2} \le \frac{2}{(n+1)^2}$$

since the numerator is bounded by 2.

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8. Find the vertices of the conic given by the equation $4x^2 - y^2 + 8x - 4y + 12 = 0$

Answer. Complete the square:

$$4(x^2+2x+1) - (y^2+4y+4) + 12 - 4 + 4 = 0 \quad \text{or} \quad 4(x+1)^2 - (y+2)^2 = -12 .$$

This gives the standard form

$$-\frac{(x+1)^2}{3} + \frac{(y+2)^2}{12} = 1 .$$

This is a hyperbola with center at (-1,-2) and axis the line $x = -1$. Setting $x = -1$ gives the y coordinates of the vertices:

$$\frac{(y+2)^2}{12} = 1 \quad \text{or} \quad y = -2 \pm \sqrt{12} .$$

Thus the vertices are at $(-1, -2-2\sqrt{3})$ and $(-1, -2+2\sqrt{3})$.

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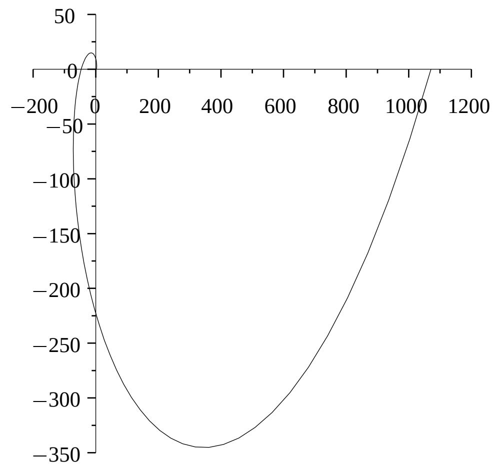
9. Find the area of the region enclosed by the curve given in polar coordinates by $r = 2e^{\theta},\ 0 \le \theta \le 2\pi$ and the segment of the x axis between $x = 2$ and $x = 2e^{2\pi}$.

PSfrag replacements



Answer. From the diagram we see that the area is

$$Area = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = 2\int_0^{2\pi} e^{2\theta} d\theta = e^{2\theta}\Big|_0^{2\pi} = e^{4\pi} - 1 \ .$$

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10. a) Find the general solution of the homogeneous differential equation $y'' - 3y' + 2y = 0$.

Answer. The roots of the equation $r^2 - 3r + 2 = 0$ are 1,2. Thus the general solution is

$$y_h = Ae^x + Be^{2x} \ .$$

b) Find a particular solution of the homogeneous differential equation $y'' - 3y' + 2y = \sin x$.

Answer. We use the method of undetermined coefficients. Try a solution of the form $y = A\cos x + B\sin x$:

$$(-A\cos x - B\sin x) - 3(-A\sin x + B\cos x) + 2(A\cos x + B\sin x) = \sin x \ ,$$

leading to the equations $-A - 3B + 2A = 0, \ -B - +3A + 2B = 1$. The solutions are $B = 1/10, \ A = 3/10$, so a particular solution is

$$y_p = 0.3\cos x + 0.1\sin x \ .$$