Mathematics 1220 Calculus II, Examination 4, Answers

1. Consider the conic given by the equation \(4x^2 - 24x + 9y^2 + 18y + 9 = 0\).
   a) What kind of conic is it?

   **Solution.** Complete the square to get
   \[
   4(x^2 - 6x + 9) + 9(y^2 + 2y + 1) = -9 + 36 + 9 = 36,
   \]
   so the conic is an ellipse.

   b) Give the coordinates of its vertex/vertices.

   **Solution.** Now we put the equation in normal form:
   \[
   \frac{(x - 3)^2}{9} + \frac{(y + 1)^2}{4} = 1.
   \]
   The center of the ellipse is at \((3, -1)\), and the major radius is 3, and the minor radius is 2.
   Thus the major axis is the line \(y = -1\), and the vertices are each 3 units from the center, so are at \((6, -1)\) and \((0, -1)\).

   c) Give the coordinates of its focus/foci.

   **Solution.** The distance of the foci from the center is \(c\) where
   \[
   c^2 = a^2 - b^2 = 9 - 4 = 5.
   \]
   Thus the foci are at \((3 \pm \sqrt{5}, -1)\).

2. Consider the conic given by the equation \(4x^2 - 24x + 9y + 18 = 0\).
   a) What kind of conic is it? b) and c): where is the vertex and focus?

   **Solution.** Complete the square to get
   \[
   4(x^2 - 6x + 9) + 9y - 18 = 0 \quad \text{or} \quad y - 2 = -\frac{4}{9}(x - 3)^2,
   \]
   so the conic is a parabola with vertex at \((3, 2)\) and \(4p = -4/9\). The parabola opens downward, and is \(p\) units below the vertex, so is at \((3-(1/9), 2)\).

3. Find the equation of the hyperbola whose vertices are at \((-3, 0)\), \((3, 0)\) and which goes through the point \((6, 6)\).

   **Solution.** The center is at the origin, and the major axis is the \(x\)-axis, so the equation of the hyperbola is of the form
   \[
   \frac{x^2}{9} - \frac{y^2}{b^2} = 1.
   \]
We solve for $b$ using the fact that $(6,6)$ is on the curve:

$$\frac{36}{9} - \frac{36}{b^2} = 1 \quad \text{so} \quad \frac{36}{b^2} = 3 ,$$

and thus $b^2 = 12$. This gives the equation

$$\frac{x^2}{9} - \frac{y^2}{12} = 1 .$$

4. Find the length of the spiral $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$.

**Solution.** We start with the equation $ds^2 = dr^2 + r^2 d\theta^2$. Now, $dr = 2e^{2\theta} d\theta$, so

$$ds^2 = 4e^{4\theta} d\theta^2 + e^{4\theta} d\theta^2 = 5e^{4\theta} d\theta^2 .$$

Thus $ds = \sqrt{5}e^{2\theta} d\theta$, and the length is

$$\int_0^{2\pi} \sqrt{5}e^{2\theta} d\theta = \frac{\sqrt{5}}{2} e^{2\theta} |_0^{2\pi} = \frac{\sqrt{5}}{2} (e^{4\pi} - 1) .$$

5. Find the area enclosed by the limaçon $r = 3 + 2 \sin \theta$.

**Solution.** $dA = (1/2)r^2 d\theta = (1/2)(3 + 2 \sin \theta)^2 d\theta = (1/2)((+12 \sin \theta + 4 \sin^2 \theta)d\theta$. Thus the area is (and, in order to integrate, we have used the half-angle formula on the last term):

$$\frac{1}{2} \int_0^{2\pi} (9 + 12 \sin \theta + 2(1 - \cos 2\theta)) d\theta = 11\pi ,$$

since the integrals of the trigonometric terms is zero.