Calculus III
Exam 3, Summer 2003, Answers

Problems are worth 20 points each. You may use calculators and Tables of Integrals. You must show enough work to convince me that you know how to do the problems. 1. Find the limits:

a) $\lim_{x\to 0} x\ln x =$

Answer. L'Hôpital's rule applies to $\ln x/(1/x)$, so

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{x}{1/(\ln x)} =^{l'H} \lim_{x\to 0}\frac{(1/x)}{(-1/x^2)} = \lim_{x\to 0}(-x) = 0\ .$$

b) $\lim_{x\to\pi/2}(x-\pi/2)\tan x =$

Answer. If we replace $\tan x$ by $\sin x/\cos x$, then both numerator and denominator go to infinity, so L'Hôpital's rule applies:

$$\lim_{x\to\pi/2}(x-\pi/2)\tan x = \lim_{x\to\pi/2}\frac{(x-\pi/2)\sin x}{\cos x} =^{l'H} \lim_{x\to\pi/2}\frac{\sin x-(x-\pi/2)\cos x}{-\sin x} = -1\ .$$

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2. Find the definite integrals:

a) $\int_0^\infty xe^{-x^2}dx =$

Answer. First calculate the integral from 0 to A, using the substitution $u = x^2,\ du = 2xdx$:

$$\int_0^A xe^{-x^2}dx = \frac{1}{2}\int_0^{A^2} e^{-u}du = \frac{1}{2}(1-e^{-A^2})\ .$$

Now, since $e^{-A^2}\to 0$ as $A\to\infty$, when we take the limits we find

$$\int_0^\infty xe^{-x^2}dx = \frac{1}{2}\ .$$

b) $\int_0^2 \ln x dx =$

Answer. Since $\int \ln x dx = x\ln x - x$,

$$\int_0^2 \ln x dx = \lim_{\varepsilon\to 0}\int_\varepsilon^2 \ln x dx = 2\ln 2 - 2 - \lim_{\varepsilon\to 0}(\varepsilon\ln\varepsilon - \varepsilon)$$
$$= 2\ln 2 - 2 - \lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon - \lim_{\varepsilon\to 0}\varepsilon = 2\ln 2 - 2\ ,$$

by Problem 1, part a).

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3. Does the series converge or diverge? Give your reasoning.

a) $\sum_{n=1}^\infty \frac{n+1}{n^3}$ converges.

Answer. Since $n+1 \leq 2n$, this series converges by comparison with the p-series for $p=2$:

$$\frac{n+1}{n^3} \leq \frac{2n}{n^3} \leq \frac{2}{n^2} \, .$$

b) $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$ diverges.

Answer. Use the ratio test:

$$\frac{e^{n+1}}{(n+1)^e} \frac{n^e}{e^n} = e\Big(\frac{n}{n+1}\Big)^e \to e > 1 \, .$$

c) $\sum_{n=1}^{\infty} \frac{2^n}{(3+\frac{1}{n})^n}$ converges.

Answer. Since $3+(1/n) \geq 3$,

$$\frac{2^n}{(3+\frac{1}{n})^n} \leq \Big(\frac{2}{3}\Big)^n \, ,$$

so the series converges by comparison with the geometric series.

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4. Find the radius of convergence of the series:

a) $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$

Answer. $\frac{2^{n+1}}{n+1} \frac{n}{2^n} = 2\big(\frac{n}{n+1}\big) \to 2$, so the radius of convergence is 1/2.

b) $\sum_{n=1}^{\infty} \frac{2^n}{n!} x^n$

Answer. $\frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \frac{2}{n+1} \to 0$, so the radius of convergence is ∞.

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5. Find the Taylor series centered at the origin for the function

$$F(x) = \int_0^x \frac{dt}{1-t^4} \, .$$

Answer. We start with the geometric series:

$$\frac{1}{1-x} = \sum_0^{\infty} x^n \, .$$

Substitute t^4 for x to get

$$\frac{1}{1-t^4} = \sum_0^{\infty} t^{4n} \, .$$

Now integrate both sides; integrating the right hand side term by term:

$$F(x) = \int_0^x \frac{dt}{1-t^4} = \sum_0^{\infty} \frac{t^{4n+1}}{4n+1} \, .$$