# Math 1220 Calculus II, Examination 3, October 30, November 1, 2003: Solutions

Problems are worth 20 points each. You may use calculators and Tables of Integrals. You must show enough work to convince me that you know how to do the problems.

1. Find the limits:

a). $\quad \lim\limits_{t \to \infty} \dfrac{t \ln t}{1 + t^2} =$

**Solution**. L'Hôpital's rule applies, so

$$\lim_{t \to \infty} \frac{t \ln t}{1 + t^2} =^{l'H} \lim_{t \to \infty} \frac{\ln t + 1}{2t} =^{l'H} \lim_{t \to \infty} \frac{1/t}{2} = 0 \ .$$

b). $\quad \lim\limits_{x \to \pi/2} \dfrac{\cos x}{x - \frac{\pi}{2}} =$

**Solution**. L'Hôpital's rule applies, so

$$\lim_{x \to \pi/2} \frac{\cos x}{x - \pi/2} =^{l'H} \lim_{x \to \pi/2} \frac{-\sin x}{1} = -1 \ .$$

2. Find the definite integrals:

a). $\quad \displaystyle\int_0^\infty t e^{-t} dt =$

First we find the indefinite integral, integrating by parts: Let $u = t$, $du = dt$, $dv = e^{-t} dt$, $v = -e^{-t}$:

$$\int t e^{-t} dt = -t e^{-t} - e^{-t} \ ,$$

so

$$\int_0^A t e^{-t} dt = (-t e^{-t} - e^{-t})\big|_0^A = -A e^{-A} - e^{-A} - (0 - 1) \to 1$$

as $A \to \infty$.

b). $\quad \displaystyle\int_0^2 \dfrac{dx}{\sqrt{x}} =$

**Solution**. First, the indefinite integral: $\int x^{-1/2} dx = 2x^{1/2}$. So,

$$\int_\epsilon^2 \frac{dx}{\sqrt{x}} = 2x^{1/2}\big|_\epsilon^2 = 2\sqrt{2} - 2\sqrt{\epsilon} \to 2\sqrt{2}$$

as $\epsilon \to 0$.

3. Does the series converge or diverge? Give your reasoning.

a) $$\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^4 - n^2 + n} \quad \text{converges.}$$

**Solution.** Since $1 + 1/n^2 < 2$ and $n^2 - 1 + 1/n > n^2/2$ for every $n$, this series converges by comparison with the $p$-series for $p = 2$. First we divide numerator and denominator by $n^2$:

$$\frac{n^2 + 1}{n^4 - n^2 + n} = \frac{1 + 1/n^2}{n^2 - 1 + 1/n} \leq \frac{2}{n^2/2} = \frac{4}{n^2} \; .$$

b) $$\sum_{n=1}^{\infty} \frac{n!}{e^n} \quad \text{diverges.}$$

**Solution.** Use the ratio test:

$$\frac{(n+1)!}{e^{n+1}} \frac{e^n}{n!} = \frac{n+1}{e} \to \infty \; .$$

c) $$\sum_{n=1}^{\infty} \frac{e^{\cos(n\pi)}}{n^2} \quad \text{converges.}$$

**Solution.** Since $|\cos x| \leq 1$ for all $x$, $e^{\cos(n\pi)} \leq e$ for all $n$, so the series converges by comparison with the series $\sum 1/n^2$.

4. a) Let $f(x) = \sum_{n=0}^{\infty} (2^n - 1)x^n$. What is the radius of convergence of the series?

**Solution.** We use the ratio test:

$$\frac{2^{n+1} - 1}{2^n - 1} = 2 \frac{1 - 1/2^{n+1}}{1 - 1/2^n} \to 2 \; ,$$

so $R = 1/2$.

b). Write $f(x)$ in closed form (that is, as an algebraic expression). **Solution.**

$$\sum_{n=0}^{\infty} (2^n - 1)x^n = \sum_{n=0}^{\infty} (2x)^n - \sum_{n=0}^{\infty} x^n = \frac{1}{1 - 2x} - \frac{1}{1 - x}$$

$$= \frac{x}{(1 - 2x)(1 - x)} \; .$$

2

5. For the Maclaurin series expansion:

$$\frac{t}{2-t^4} = \sum_{n=0}^{\infty} a_n t^n$$

find the values of $a_0, a_1, a_2, a_3$. **Solution**.

$$\frac{t}{2-t^4} = \frac{t}{2}\frac{1}{1-t^2/2} = \frac{t}{2}\sum_0^{\infty}(\frac{t^2}{2})^n = \sum_0^{\infty}\frac{t^{2n+1}}{2^{n+1}}$$

$$= \frac{t}{2} + \frac{t^3}{4} + \frac{t^5}{8}\cdots \ ,$$

so $a_0 = 0, a_1 = .5, a_2 = 0, a_3 = .25, a_4 = 0, a_5 = .125$.