**Calculus II**
**Exam 2, Summer 2003, Answers**

Find all the integrals. Remember that definite integrals should have numerical answers. You MUST show your work.

1a. $\int x(\ln x)dx$

**Answer**. We integrate by parts to get rid of the logarithm. Let $u = \ln x,\ du = dx/x,\ dv = xdx, v = x^2/2$. The integral becomes

$$uv - \int vdu = \frac{x^2 \ln x}{2} - \int \frac{x^2}{2}\frac{dx}{x} \ .$$

The last integral is $x^2/4 + C$, so the answer is

$$\int x(\ln x)dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C \ .$$

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1b. $\int \frac{\ln(x^2)}{x}dx$

**Answer**. Resist the temptation to let $u = x^2$, and instead first note that $\ln(x^2) = 2\ln x$. Now we can make the substitution $u = \ln x,\ du = dx/x$, to get

$$\int \frac{\ln(x^2)}{x}dx = 2\int \frac{\ln x}{x}dx = 2\int udu = u^2 + C = (\ln x)^2 + C \ .$$

The substitution $u = x^2$ doesn't fail; it just makes more work. We get $du/2u = dx/x$, leading to

$$\int \frac{\ln(x^2)}{x}dx = \frac{1}{2}\int \frac{\ln u}{u}du = \frac{(\ln(x^2))^2}{4} + C \ ,$$

which is the same answer. Check it!

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2. $\int \frac{dx}{x(x-1)(x+2)}$

**Answer**. We use partial fractions. We set

$$\frac{1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} = \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)}$$

Set $x$ equal to the roots and equate numerators:

$$x = 0: \quad -2A = 1\ , \quad x = 1: \quad 3B = 1\ , \quad x = -2: \quad C(-2)(-3) = 1\ ,$$

so $A = -1/2, B = 1/3, C = -1/6$. This gives the answer as

$$-1/2\int \frac{dx}{x} + 1/3\int \frac{dx}{x-1} - 1/6\int \frac{dx}{x+2} = -\frac{\ln x}{2} + \frac{\ln(x-1)}{3} - \frac{\ln(x+2)}{6} + C \ .$$

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3. $\int_0^2 \frac{e^x}{1+e^{2x}}dx$

**Answer**. Let $u = e^x,\ du = e^x dx$. For $x = 0$, we have $u = 1$, and for $x = 2,\ u = e^2$. The integral is

$$\int_1^{e^2} \frac{du}{1+u^2} du = \arctan u\Big|_1^{e^2} = \arctan(e^2) - \frac{\pi}{4} = .65088\ .$$

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4. $\int x(x+1)^{12} dx$

**Answer**. There are lots of ways to do this.

A. Use the substitution $u = x+1$, so that $x(x+1)^{12}dx = (u-1)u^{12}du = (u^{13} - u^{12})du$. Then the integral is

$$\int (u^{13} - u^{12})du = \frac{(x+1)^{14}}{14} - \frac{(x+1)^{13}}{13} + C\ .$$

B. Let $x = (x+1) - 1$, so that $x(x+1)^{12} = (x+1)^{13} - (x+1)^{12}$, and we get the same thing.

C. Integrate by parts to get rid of the $x$ term: $u = x, du = dx, dv = (x+1)^{12}dx, v = (x+1)^{13}/13$, leading to

$$\int x(x+1)^{12}dx = \frac{1}{13}\left[x(x+1)^{13} - \int (x+1)^{13}dx\right] = \frac{1}{13}\left[x(x+1)^{13} - \frac{(x+1)^{14}}{14}\right] + C\ .$$

You can check that these are the same answers, and can both be rewritten as

$$\int x(x+1)^{12}dx = \frac{(x+1)^{13}(13x-1)}{182} + C\ .$$

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5. $\int_2^4 \frac{dx}{x(x-1)^2}$

**Answer**. We use partial fractions. We set

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)(x+2)}$$

Setting $x = 0$, we obtain $A = 1$, at $x = 1$ we obtain $C = 1$. Now compare the coefficient of $x^2$ on both sides, to obtain $0 = A + B$, so $B = -1$. Thus the integral is

$$\int_2^4 \left(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right)dx = (\ln x - \ln(x-1) - (x-1)^{-1})\Big|_2^4$$

$$= \ln 4 - \ln 3 - \frac{1}{3} - (\ln 2 - \ln 1 - 1) = \ln\frac{2}{3} + \frac{2}{3} = 2.612\ .$$