1220-90 Exam 2

Summer 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. A correct answer without accompanying work may not receive full credit. Please try to do all all work in the space provided and circle your final answer.

1. (16pts) Evaluate the following limits using L'Hôpital's rule or any other method.

(a) (4pts) $\lim_{x\to 0} \frac{x}{\cos x} = \frac{0}{1} = 0$ 1/4

4

(b) (4pts) $\lim_{x\to 0} \frac{x\cos x}{\tan(2x)} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x - x\sin x}{\sec^2(2x)\cdot 2} = \frac{1-0}{1\cdot 2} = \frac{1}{2}$

4

(c) (4pts) $\lim_{x\to 0^+} x\ln x$ **Hint:** Rewrite as a fraction of indeterminate type $\frac{\infty}{\infty}$

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$

4

(d) (4pts) $\lim_{x\to 0} \frac{\sin x - x}{x^3} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{-\sin x}{6x}$

$\overset{L'H}{=} \lim_{x\to 0} \frac{-\cos x}{6} = \frac{-1}{6}$

4

2. (8pts) Consider the sequence with first five terms given by

$$a_1 = 1 \qquad a_2 = \frac{2}{3} \qquad a_3 = \frac{4}{9} \qquad a_4 = \frac{8}{27} \qquad a_5 = \frac{16}{81}$$

Assuming the sequence continues on in this same manner, answer the following questions:

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(a) (2pts) Find a formula for a_n. $a_n = (2/3)^{n-1}$ /

(b) (2pts) $\lim_{n\to\infty} a_n =$ 0.

(c) (2pts) Is this sequence convergent or divergent? convergent

(d) (2pts) $\sum_{n=1}^{\infty} a_n =$ 3 (write 'diverges' if the series diverges)

Geometric series w/ $a=1$, $r=\frac{2}{3}<1$. Converges to $\frac{1}{1-\frac{2}{3}} = 3$

3. (11pts) In this problem, you will compute the value of an improper integral in two steps.

(a) (7pts) Use integration by parts to compute the definite integral

$$\int_0^t xe^{-x}\,dx.$$

Note: Your answer should be a function of t.

7

$$\int_0^t xe^{-x}dx = \left(-xe^{-x}\Big|_0^t + \int_0^t e^{-x}dx\right.$$

3 $u = x \quad du = dx$

$dv = e^{-x}dx \quad v = -e^{-x}$

$$= -te^{-t} + \left(-e^{-x}\Big|_0^t\right.$$

$$= -te^{-t} - e^{-t} + 1$$

(b) (4pts) Take the limit as $t \to \infty$ of your answer in part (a) to find

$$\int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx.$$

4

$$\lim_{t\to\infty}\left(-te^{-t} - e^{-t} + 1\right) = 1.$$

can use L'Hopital on this term.

4. (15pts) All of the following series are convergent. Evaluate them.

(a) (5pts) $\sum_{n=1}^{\infty} 2\left(\frac{1}{7}\right)^{n-1}$ Geometric series w/ $a = 2$ + $r = \frac{1}{7} < 1$.

5

Converges to $\frac{2}{1-\frac{1}{7}} = \frac{2}{6/7} = \frac{14}{6} = \frac{7}{3}$

$\frac{a}{1-r}$

(b) (5pts) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3^{n-1}} = -1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \cdots$

5

Geometric series with $a = -1$ and $r = -1/3$

Converges to $\frac{-1}{1-(-1/3)} = -3/4$

(c) (5pts) $\sum_{n=1}^{\infty}\left(\frac{3}{n+1} - \frac{3}{n+2}\right)$

5

$$S_N = \sum_{n=1}^{N}\left(\frac{3}{n+1} - \frac{3}{n+2}\right) = \left(\frac{3}{2} - \cancel{\frac{3}{3}}\right) + \left(\cancel{\frac{3}{3}} - \cancel{\frac{3}{4}}\right) + \left(\cancel{\frac{3}{4}} - \cancel{\frac{3}{5}}\right) + \cdots + \left(\cancel{\frac{3}{N+1}} - \frac{3}{N+2}\right)$$

$$= \frac{3}{2} - \frac{3}{N+2}.$$

$$\sum_{n=1}^{\infty}\left(\frac{3}{n+1} - \frac{3}{n+2}\right) = \lim_{N\to\infty} S_N \overset{2}{=} \lim_{N\to\infty} \frac{3}{2} - \frac{3}{N+2} = \frac{3}{2}$$

5. (20pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent. You do not need to show work.

20

2pts each

C D $\sum_{n=1}^{\infty} \frac{1}{n}$

C D $\sum_{n=1}^{\infty} \frac{1}{1+x^2}$

C D $\sum_{n=1}^{\infty} \frac{7}{3^n}$

C D $\sum_{n=1}^{\infty} \frac{n^2}{4n^2+9}$

C D $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+1}}$

C D $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$

C D $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

C D $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$

C D $\sum_{n=1}^{\infty} (-1)^{n-1} n$

C D $\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{n}}$

6. (10pts) Find the first four terms of the power series representation

$$f(x) = \frac{1}{(1-x/2)^2} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

10

$c_0 =$ 1 $c_1 =$ 1 $c_2 =$ 3/4 $c_3 =$ 1/2

What is the radius of convergence of this power series? 2

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{if } |x| < 1.$$

$$\frac{1}{(1-x)^2} = D_x\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad \text{if } |x| < 1.$$

$$\frac{1}{(1-x/2)^2} = 1 + 2\left(\frac{x}{2}\right) + 3\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right)^3 + \cdots \quad \text{if } \left|\frac{x}{2}\right| < 1$$

$$= 1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \cdots \quad \text{or } |x| < 2.$$

7. (10pts) Determine whether the following series is absolutely convergent, conditionally convergent, or divergent. Show how you reached your conclusion.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+1}$$

First check abs. convergence: look at $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Use Integral Test:

10

$$\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \frac{1}{2}\int_2^{t^2+1} \frac{1}{u}\,du$$

$u = x^2+1$
$du = 2x\,dx$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\ln(t^2+1) - \ln 2\right) = \infty.$$

So not abs. convergent.

Check for convergence using AST:

If $f(x) = \frac{x}{x^2+1}$, then $f'(x) = \frac{1-x^2}{(x^2+1)^2} \le 0$ when $x \ge 1$.

So series is alternating and terms are getting smaller (in abs. value).

So converges by AST.

8. (10pts) Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{3^n (x-1)^n}{n!}$$

3

10

Use Ratio Test w/ $a_n = \frac{3^n |x-1|^n}{n!}$

3

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}|x-1|^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n |x-1|^n} = \frac{3|x-1|}{n+1} \to 0$$

for any x as $n \to \infty$.

So radius of convergence is ∞,

and interval of convergence is $(-\infty, \infty)$