1220-90 Exam 2
Spring 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. A correct answer without accompanying work may not receive full credit. Please try to do all all work in the space provided. Please circle your final answer.

1. (15pts) Evaluate the following limits using L'Hôpital's rule or any other method. Note, the correct answer may be $\pm\infty$.

   (a) (5pts) $\lim_{x\to\infty} \frac{2x^2-1}{x^2-5x+1}$ $= \lim_{x\to\infty} \frac{x^2(2-1/x^2)}{x^2(1-5/x+1/x^2)} = \lim_{x\to\infty} \frac{2-1/x^2}{1-5/x+1/x^2} = 2.$

   or L'Hopitals.

   5

   (b) (5pts) $\lim_{x\to 0} \frac{x^2}{\sin x \cos x}$ $\underset{LH}{=} \lim_{x\to 0} \frac{2x}{\cos^2 x - \sin^2 x} = \frac{0}{1} = 0.$

   5

   (c) (5pts) $\lim_{x\to\infty} \frac{x^2}{(\ln x)^2}$ $\underset{LH}{=} \lim_{x\to\infty} \frac{2x}{2(\ln x)\frac{1}{x}} = \lim_{x\to\infty} \frac{x^2}{\ln x}$ $\underset{LH}{=} \lim_{x\to\infty} \frac{2x}{\frac{1}{x}} = \lim_{x\to\infty} 2x^2 = +\infty$

   5

2. (12pts) Evaluate the following improper integrals. Write 'divergent' if the integral diverges.

   (a) (6pts) $\int_1^\infty \frac{2}{x^5}\,dx$ $= \lim_{t\to\infty} \int_1^t \frac{2}{x^5}\,dx$

   $= \lim_{t\to\infty} \left(-\frac{1}{2}x^{-4}\Big|_1^t\right) = \lim_{t\to\infty}\left(\frac{-1}{2t^4}+\frac{1}{2}\right) = \frac{1}{2}$

   6

   (b) (6pts) $\int_1^\infty \frac{1}{x^2+1}\,dx$ $= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx$

   $= \lim_{t\to\infty} \left(\tan^{-1} x\Big|_1^t\right)$

   $= \lim_{t\to\infty} \tan^{-1} t - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

   6

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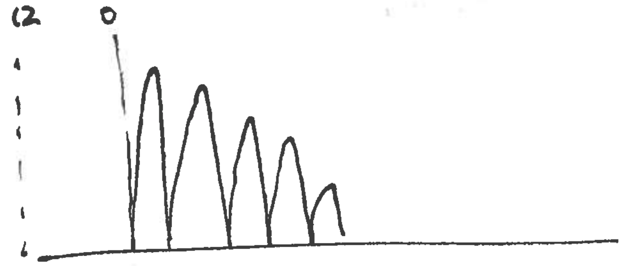
3. (13pts) A rubber ball will bounce back up to two-thirds of the height from which it was dropped. Suppose the ball is dropped from an initial height of 12 feet and allowed to bounce until it comes to rest. Let a_n denote the maximum height attained by the ball after the nth time it hits the floor.



3

(a) (3pts) Find a_1, a_2, and a_3.

$a_1 =$ 8

$a_2 =$ 16/3

$a_3 =$ 32/9

2

(b) (2pts) Find a formula for a_n.

$$a_n = (12)\left(\tfrac{2}{3}\right)^n$$

2

(c) (2pts) $\lim_{n\to\infty} a_n =$ 0

6

(d) (6pts) How far has the ball traveled (both up and down) from when it was dropped to when it comes to a rest? (Compute the total distance traveled after an infinite number of bounces using a series)

$$12 + \sum_{n=1}^{\infty} 2(12)\left(\tfrac{2}{3}\right)^n$$ ← geometric series $a=16$ $r=\frac{2}{3}$

$$= 12 + 16 + 16\left(\tfrac{2}{3}\right) + 16\left(\tfrac{2}{3}\right)^2 + \cdots$$

$$= 12 + \frac{16}{1-\frac{2}{3}} = 12 + 16\cdot 3 = 60\text{ ft}$$

4. (10pts) Both of the series below are convergent. Evaluate them.

5

(a) (5pts) $\sum_{n=1}^{\infty}\left(\frac{2}{9}\right)^n = \left(\frac{2}{9}\right) + \left(\frac{2}{9}\right)^2 + \left(\frac{2}{9}\right)^3 + \cdots$

$$= \frac{2/9}{1-2/9} = \frac{2/9}{7/9} = \frac{2}{7}$$

5

(b) (5pts) $\sum_{n=1}^{\infty}\left(\frac{2}{n+2} - \frac{2}{n+3}\right)$

$$S_N = \sum_{n=1}^{N}\left(\frac{2}{n+2} - \frac{2}{n+3}\right) = \left(\frac{2}{3} - \frac{2}{4}\right) + \left(\frac{2}{4} - \frac{2}{5}\right) + \cdots \left(\frac{2}{N+2} - \frac{2}{N+3}\right)$$

$$= \frac{2}{3} - \frac{2}{N+3}.$$

$$\lim_{N\to\infty} \frac{2}{3} - \frac{2}{N+3} = \frac{2}{3}$$

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7. (10pts) Determine whether the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)\ln(n+1)}$$

Note that the series is alternating, and for $n \geq 1$,

$$\frac{1}{(n+1)\ln(n+1)}$$

is positive and decreasing, so series is convergent. 4

Check for absolute convergence: examine $\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$

3

using integral test

$$\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{(x+1)\ln(x+1)}\,dx$$

$u = \ln(x+1)$

$du = \frac{1}{x+1}\,dx$

$$= \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t+1)} u^{-1}\,du = \lim_{t\to\infty} \left(\ln u \Big|_{\ln(2)}^{\ln(t+1)}\right) = \lim_{t\to\infty} \left(\ln(\ln(t+1)) - \ln(\ln\right.$$

$= +\infty$.

not absolutely conv.

3

8. (10pts) Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{5^{n-1} n^3}$$

$$a_n = \frac{|x|^{n-1}}{5^{n-1} n^3}$$

5

Use Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x|^n}{5^n (n+1)^3} \cdot \frac{5^{n-1} n^3}{|x|^{n-1}} = \lim_{n\to\infty} \left(\frac{n^3}{(n+1)^3}\right) \frac{|x|}{5}$$

$$= \frac{|x|}{5}$$

So abs. convergent if $\frac{|x|}{5} < 1 \Rightarrow |x| < 5$. 2

Check endpoints:

$x = 5$ $\quad \sum_{n=1}^{\infty} \frac{5^{n-1}}{5^{n-1} n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ convergent.

3

$x = -5$ $\quad \sum_{n=1}^{\infty} \frac{(-5)^{n-1}}{5^{n-1} n^3} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ convergent.

$[-5, 5]$

5. (20pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent. No need to show work.

20

2pts each

C D $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

C D $\sum_{n=1}^{\infty} \frac{n}{n+5}$

C D $\sum_{n=1}^{\infty} \left(\frac{\pi}{2}\right)^{n-1}$

C D $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

C D $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

C D $\sum_{n=1}^{\infty} \frac{3}{n^2+4n+9}$

C D $\sum_{n=1}^{\infty} \frac{n!}{3^n}$

C D $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

C D $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

C D $\sum_{n=1}^{\infty} (-1)^{n-1} e^n$

6. (10pts) Find the first four terms of the power series representation

$$f(x) = \frac{1}{(1+x)^2} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

2 $c_0 =$ 1

2 $c_1 =$ -2

2 $c_2 =$ 3

2 $c_3 =$ -4

What is the radius of convergence of this power series? 1 2

Something partially correct (+5)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \qquad |x|<1$$

$$\frac{1}{(1-x)^2} = D_x\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \qquad |x|<1$$

$$\frac{1}{(1+x)^2} = \frac{1}{(1-(-x))^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots \qquad |x|<1$$