1220-90 Exam 1

Summer 2014

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Please circle your final answer. The last page contains some useful identities.

1. (15 pts) Find the indicated derivatives. No need to simplify.

   (a) (5pts) $D_x(\ln(\tan x))$

   $$= \frac{1}{\tan x}\sec^2 x = \frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x} = \frac{1}{\sin x\cos x}.$$

   5

   (b) (5pts) $D_x(e^{\sin(7x)})$

   $$= e^{\sin(7x)}\cos(7x)\cdot 7$$

   5

   (c) (5pts) $D_x(x^{5x}) = D_x(e^{5x\ln x})$

   $$= e^{5x\ln x}\left(5\ln x + 5x\left(\tfrac{1}{x}\right)\right) = x^{5x}(5\ln x + 5)$$

   5

-1 missing +C.

2. (10pts) Find the indicated antiderivatives. Remeber $+C$!!

   (a) (5pts) $\int e^x \cosh(e^x)\, dx = \int \cosh(u)\, du = \sinh(u) + C$

   $u = e^x$

   $du = e^x dx$

   $= \sinh(e^x) + C$

   5

   (b) (5pts) $\int \frac{1}{2-3x}\, dx = -\frac{1}{3}\int \frac{1}{u}\, du = -\frac{1}{3}\ln|u| + C$

   $u = 2-3x$

   $du = -3dx$

   $= -\frac{1}{3}\ln|2-3x| + C.$

   5

25

3. (4pts) Suppose the function f is one-to-one. Listed below are a few values of f and its derivative:

x	$f(x)$	$f'(x)$
0	1	3
1	2	4
2	4	5

For example, $f(0) = 1$, $f'(1) = 4$, etc. Fill in the blanks:

2 $f^{-1}(2) =$ $\underline{1}$

2 $(f^{-1})'(2) =$ $\underline{\frac{1}{f'(1)}} = \frac{1}{4}$.

4. (4pts) Find a formula for the inverse of the function $f(x) = x^2 + 2x + 5$, $x > -1$.

4

$$x = y^2 + 2y + 5$$
$$= (y^2 + 2y + 1) + 4$$
$$= (y+1)^2 + 4.$$
$$x - 4 = (y+1)^2$$
$$y = \sqrt{x-4} - 1$$

5. (10pts) Evaluate the following. Any answer representing an angle should be given in radians.

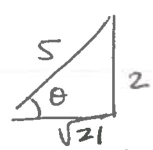
2 (a) $\sin^{-1}(\frac{\sqrt{2}}{2}) =$ $\underline{\pi/4}$

2 (b) $\cos^{-1}(0) =$ $\underline{\pi/2}$

2 (c) $\sin(\sin^{-1}(\frac{1}{2})) =$ $\underline{1/2}$

2 (d) $\sin^{-1}(\sin(3\pi)) =$ $\underline{0}$

2 (e) $\cos(\sin^{-1}(\frac{2}{5})) =$ $\underline{\sqrt{21}/5}$



6. (10pts) A 100 gram sample of a radioactive isotope was found to only contain 73 grams of radioactive material after 17 days. Let $A(t)$ denote the amount of radioactive material (in grams) after t days. So $A(0) = 100$ and $A(17) = 73$.

(a) (6pts) Find constants C and k such that $A(t) = Ce^{kt}$. No need to simplify.

3 $C =$ $\underline{100}$

$$100 = A(0) = Ce^{k \cdot 0} = C.$$

3 $k =$ $\underline{\frac{1}{17}\ln\left(\frac{73}{100}\right)}$

$$73 = A(17) = 100\, e^{k(17)}$$
$$\ln\left(\frac{73}{100}\right) = 17k \implies k = \frac{1}{17}\ln\left(\frac{73}{100}\right)$$

(b) (4pts) Find the half-life of the isotope; that is, the amount of time (in days) that it takes for half of the isotope to decay. No need to simplify.

4

$$50 = 100\, e^{\frac{t}{17}\ln(73/100)}$$
$$\frac{1}{2} = e^{\frac{t}{17}\ln(73/100)}$$
$$\ln\left(\frac{1}{2}\right) = \frac{t}{17}\ln(73/100) \implies t = \frac{17\ln(1/2)}{\ln(73/100)}$$

28

7. (25pts) Evaluate the following indefinite integrals. Remember $+C$!!

(a) $\int \cos^3 x \sin^2 x \, dx$ $= \int \cos x \left(1-\sin^2 x\right) \sin^2 x \, dx = \int (u^2-u^4)\,du$

$u = \sin x$

$du = \cos x \, dx$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$

$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

5

(b) $\int x e^{3x} \, dx$ $= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x} dx$

$u = x \qquad du = dx$

$dv = e^{3x} dx \qquad v = \frac{1}{3}e^{3x}$

$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

5

(c) $\int \sin^{-1} x \, dx$ **Hint:** Use integration by parts with $u = \sin^{-1} x$ and $dv = dx$.

$u = \sin^{-1} x \qquad du = \frac{1}{\sqrt{1-x^2}} dx$

$dv = dx \qquad v = x$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$ $\rightarrow$ $u = 1-x^2$, $du = -2x\,dx$

$= x\sin^{-1}x + \frac{1}{2}\int u^{-1/2} du$

$= x\sin^{-1}x + u^{1/2} + C = x\sin^{-1}x + \sqrt{1-x^2} + C$

5

(d) $\int \sec x \tan^3 x \, dx$ $= \int \sec x \tan x \tan^2 x$

$= \int \sec x \tan x \left(\sec^2 x - 1\right) dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

$= \int (u^2-1)\,du = \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

5

(e) $\int \cos^2 x \sin^2 x \, dx$

$= \frac{1}{4}\int (1+\cos(2x))(1-\cos(2x))\,dx = \frac{1}{4}\int \left(1-\cos^2(2x)\right) dx$

$= \frac{1}{4}\int \left(1 - \frac{1}{2}\left(1+\cos(4x)\right)\right) dx$

$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos(4x)\right) dx$

$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$

5

25

8. (12pts) Evaluate the following indefinite integral using **rationalizing substitution** in three steps.

$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx$$

Should be $\int \frac{\sqrt{1+x^2}}{x^4}\,dx$

(a) (4pts) First, write as an integral with respect to t by using the rationalizing substitution $x = \tan t$.

+6 automatically

$x = \tan t \implies dx = \sec^2 t\, dt$

$\sqrt{1+x^2} = \sec t.$

$$\int \frac{\sqrt{1+x^2}}{x^4}\,dx = \int \frac{\sec^3 t}{\tan^4 t}\,dt$$

(b) (4pts) Second, evaluate the integral from part (a). You answer should be in terms of t. **Hint:** Write the integrand in terms of $\sin t$ and $\cos t$ using the identities $\tan t = \frac{\sin t}{\cos t}$ and $\sec t = \frac{1}{\cos t}$.

12

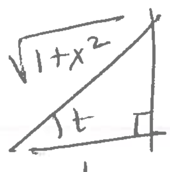
$$\int \frac{\sec^3 t}{\tan^4 t}\,dt = \int \left(\frac{1}{\cos^3 t}\right)\left(\frac{\cos^4 t}{\sin^4 t}\right)dt = \int \frac{\cos t}{\sin^4 t}\,dt$$

$u = \sin t \implies du = \cos t\, dt$

$$= \int u^{-4}\,du = -\frac{1}{3}u^{-3} + C = -\frac{1}{3}\sin^{-3} t + C$$

(c) (4pts) Finally, write you answer to part (b) in terms of x.

$\frac{x}{1} = \tan t$



$\sin t = \frac{x}{\sqrt{1+x^2}}$

$$-\frac{1}{3}\left(\frac{1}{\sin^3 t}\right) + C = -\frac{1}{3}\,\frac{(1+x^2)^{3/2}}{x^3} + C$$

9. (10pts) Use partial fractions to find $\int \frac{7}{x^2-3x-10}\,dx.$

$$\frac{7}{x^2-3x-10} = \frac{A}{x-5} + \frac{B}{x+2} = \frac{A(x+2)+B(x-5)}{(x-5)(x+2)} \implies 7 = A(x+2) + B(x-5)$$

3

Plug in $x=5$: $7 = 7A \implies A = 1.$

plug in $x=-2$: $7 = -7B \implies B = -1.$ 3

10

$$\int \frac{7}{x^2-3x-10}\,dx = \int\left(\frac{1}{x-5} - \frac{1}{x+2}\right)dx$$ 4

$$= \ln|x-5| - \ln|x+2| + C$$

22