1220-90 Exam 1

Spring 2013

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Please circle your final answer. The last page contains some useful identities.

1. (20pts) Find the indicated derivatives

   (a) (5pts) $D_x(e^{5x})$

   $5e^{5x}$

5

   (b) (5pts) $D_x(\sinh(\ln x))$

   $\cosh(\ln x) \cdot \frac{1}{x}$

5

   (c) (5pts) $D_x(\log_2(\sin^{-1}(x)))$

   $\frac{1}{\ln 2 (\sin^{-1}(x))} \cdot \frac{1}{\sqrt{1-x^2}}$

5

   (d) (5pts) $D_x(x^{7x})$

   $x^{7x} = (e^{\ln x})^{7x} = e^{7x \ln x}$

5

   $D_x(x^{7x}) = D_x(e^{7x\ln x}) = e^{7x \ln x}(7\ln x + 7)$

   $= x^{7x}(7\ln x + 7)$

2. (5pts) Suppose the function f is one-to-one. Listed below are a few values of f and its derivative:

x	$f(x)$	$f'(x)$
0	3	−2
3	5	6

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

5

Fill in the blank: $(f^{-1})'(3) =$ $-\frac{1}{2}$.

$f^{-1}(3) = 0.$

$\frac{1}{f'(0)} = -\frac{1}{2}$

25

4

3. (11pts) Consider the function

$$f(x) = \frac{x}{1-x}, \qquad x > 1.$$

Note that the domain of f is only numbers greater than 1.

(a) (4pts) Show that $f(x)$ has an inverse. **Hint:** Conclude something from the derivative.

$$f'(x) = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2} > 0 \text{ when } x > 1.$$

4

(b) (4pts) Find a formula for $f^{-1}(x)$.

$$y = \frac{x}{1-x}$$

$$x = \frac{y}{1-y} \Rightarrow x - xy = y \Rightarrow x = (1+x)y \qquad y = \frac{x}{1+x}$$

3

(c) (3pts) What is the domain of $f^{-1}(x)$?

domain $f^{-1}(x)$ = range f = $(-\infty, -1)$

4. (10pts) The half-life of Sodium-22 is 2.6 years (in other words, after 2.6 years, a given sample will contain half as much Sodium-22 as it did initially). Suppose a test tube contains 10 grams of Sodium-22. Let $S(t)$ denote the number of grams of Sodium-22 in the test tube t years in the future.

(a) (6pts) Find constants C and k such that $S(t) = Ce^{kt}$. Write your answers in the provided blanks. No need to simplify.

$C =$ 10

$k =$ $\ln(1/2)/2.6$

$$\frac{1}{2} = e^{k(2.6)} \Rightarrow k = \frac{\ln(\frac{1}{2})}{2.6} = -\frac{\ln 2}{2.6}.$$

$$C = 10$$

6

(b) (4pts) How long will it take before there is only 1 gram of Sodium-22 in the test tube? No need to simplify.

$$1 = 10e^{\left(\frac{\ln(1/2)}{2.6}\right)t}$$

$$\frac{1}{10} = e^{\left(\frac{\ln(1/2)}{2.6}\right)t} \qquad t = \frac{2.6}{\ln(\frac{1}{2})}\ln\left(\frac{1}{10}\right)$$

$$= \frac{2.6\ln(10)}{\ln 2}$$

4

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5. (18pts) Evaluate the following definite integrals using either substitution or partial fractions:

(a) (9pts) $\displaystyle\int_4^5 \frac{x-2}{x^2-4x+3}\,dx$

9

$u = x^2 - 4x + 3$

$du = 2x - 4\,dx \Rightarrow (x-2)\,dx = \frac{1}{2}\,du$

$$\int_4^5 \frac{x-2}{x^2-4x+3}\,dx = \frac{1}{2}\int_3^8 \frac{du}{u}$$

$$= \frac{1}{2}\left(\ln|u|\right)\Big|_3^8$$

$$= \frac{1}{2}\left(\ln 8 - \ln 3\right)$$

or using partial fractions

$$\frac{x-2}{x^2-4x+3} = \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x-3}$$

$\rightarrow$

$$\int_4^5 \frac{x-2}{x^2-4x+3}\,dx = \frac{1}{2}\left(\ln|x-1| + \ln|x-3|\right)\Big|_4^5$$

$$= \frac{1}{2}\left(\ln 4 + \ln 2 - \ln 3\right)$$

(b) (9pts) $\displaystyle\int_4^5 \frac{1}{x^2-4x+3}\,dx$

$$\frac{1}{x^2-4x+3} = \frac{A}{x-3} + \frac{B}{x-1} = \frac{A(x-1) + B(x-3)}{(x-3)(x-1)}$$

plug in $x=1$ $\quad -2B = 1 \Rightarrow B = -\frac{1}{2}$.

plug in $x=3$ $\quad 2A = 1 \Rightarrow A = \frac{1}{2}$.

9

$$\int_4^5 \frac{1}{x^2-4x+3}\,dx = \frac{1}{2}\int_4^5 \frac{dx}{x-3} - \frac{1}{2}\int_4^5 \frac{dx}{x-1}$$

$$= \frac{1}{2}\left(\ln|x-3|\right)\Big|_4^5 - \frac{1}{2}\left(\ln|x-1|\right)\Big|_4^5$$

$$= \frac{1}{2}\left(\ln 2 - \ln 1\right) - \frac{1}{2}\left(\ln 4 - \ln 3\right)$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 3$$

6. (18pts) Evaluate the following indefinite integrals. Remember +C!

(a) (9pts) $\int \sec^4 x \tan^3 x \, dx$

$= \int \sec^3 x \, (\tan^2 x) \sec x \tan x$

$= \int \sec^3 x \, (\sec^2 x - 1) \sec x \tan x$

9

$u = \sec x$

$du = \sec x \tan x$

$= \int (u^5 - u^3) \, du = \frac{1}{6} u^6 - \frac{1}{4} u^4 + C$

$= \frac{1}{6} \sec^6 x - \frac{1}{4} \sec^4 x + C$

or

$\int \sec^4 x \tan^3 x \, dx$

$= \int (\sec^2 x)(\tan^3 x)(\sec^2 x) dx$

$= \int (1 + \tan^2 x)(\tan^3 x)(\sec^2 x) \, dx$

$u = \tan x$

$du = \sec^2 x$

$\int u^3 + u^5 \, du$

$= \frac{1}{4} u^4 + \frac{1}{6} u^6 + C$

$= \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C$

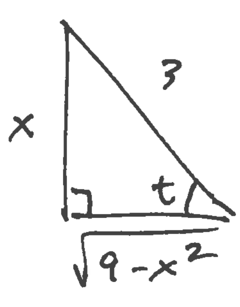
(b) (9pts) $\int \sqrt{9 - x^2} \, dx$ **Hint:** make the rationalizing substitution $x = 3 \sin t$.

$x = 3 \sin t$ $\quad \sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2 t} = 3 \cos t$

$dx = 3 \cos t \, dt$

9 $\quad \int \sqrt{9 - x^2} \, dx = \int 9 \cos^2 t \, dt = 9 \int \frac{1}{2} (1 + \cos 2t) \, dt$

$= \frac{9}{2} \int (1 + \cos 2t) \, dt$



$= \frac{9}{2} t + \frac{9}{4} \sin(2t) + C$

$\sin t = \frac{x}{3}$

$\cos t = \frac{\sqrt{9-x^2}}{3}$

$t = \sin^{-1}\left(\frac{x}{3}\right)$

$= \frac{9}{2} t + \frac{9}{2} \sin t \cos t + C$

$= \frac{9}{2} t + \frac{1}{2} x \sqrt{9 - x^2} + C$

$\left(\frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) + \frac{1}{2} x \sqrt{9 - x^2} + C \right)$

7. (18pts) Use integration by parts to find the following indefinite integrals. Remember +C!

(a) (9pts) $\int x^2 \sin(3x)\, dx$

$u = x^2 \qquad du = 2x$

$dv = \sin(3x) \qquad v = -\frac{1}{3}\cos(3x)$

9

$$\int x^2 \sin(3x)\,dx = -\frac{x^2}{3}\cos(3x) + \frac{2}{3}\int x\cos(3x)\,dx$$

$u = x \qquad du = dx$

$dv = \cos(3x) \qquad v = \frac{1}{3}\sin(3x)$

$$= -\frac{x^2}{3}\cos(3x) + \frac{2}{3}\left(\frac{x}{3}\sin(3x) - \frac{1}{3}\int \sin(3x)\,dx\right)$$

$$= -\frac{x^2}{3}\cos(3x) + \frac{2x}{9}\sin(3x) - \frac{2}{9}\left(-\frac{1}{3}\cos(3x)\right) + C$$

$$= -\frac{x^2}{3}\cos(3x) + \frac{2x}{9}\sin(3x) + \frac{2}{27}\cos(3x) + C$$

(b) (9pts) $\int \tan^{-1}(x)\, dx$ **Hint:** Set $u = \tan^{-1}(x)$ and $dv = 1\, dx$.

$u = \tan^{-1}x \qquad du = \frac{1}{1+x^2}\,dx$

$dv = dx \qquad v = x$

9

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$$

$u = 1+x^2$

$du = 2x\,dx$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{du}{u}$$

$$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$$

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Trigonometric Formulas:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin 2x = 2 \sin x \cos x$$
$$\sin x \cos y = \frac{1}{2}\big(\sin(x - y) + \sin(x + y)\big)$$
$$\sin x \sin y = \frac{1}{2}\big(\cos(x - y) - \cos(x + y)\big)$$
$$\cos x \cos y = \frac{1}{2}\big(\cos(x - y) + \cos(x + y)\big)$$

Inverse Trigonometric Formulas:

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$
$$D_x \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$
$$D_x \tan^{-1} x = \frac{1}{1 + x^2}$$
$$D_x \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}} \qquad |x| > 1$$

Hyperbolic Trigonometric Forumlas:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x}$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$D_x \sinh x = \cosh x$$
$$D_x \cosh x = \sinh x$$