1220-90 Exam 1
Fall 2012

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Please try to do all all work in the space provided. Please circle your final answer. The last page contains some useful identities.

1. (20pts) Find the indicated derivatives

4 pts each
small mistake -1
wrong, but right idea -2
something is right -3
totally incorrect -4
. no simplification required.

(a) $D_x(\ln(x^2+1))$ $= \frac{1}{x^2+1} \cdot (2x) = \frac{2x}{x^2+1}$

(b) $D_x(e^{\sqrt{x}})$

$D_x\left(e^{x^{1/2}}\right) = e^{x^{1/2}} \cdot \frac{1}{2}x^{-1/2} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

(c) $D_x(\log_2(\tan x))$ $= \frac{1}{\tan x \cdot \ln 2} \sec^2 x = \frac{\cot x \sec^2 x}{\ln 2}$

$= \frac{1}{\sin x \cos x \ln 2}$

(d) $D_x(3^{\cosh x})$

$= 3^{\cosh x}(\ln 3)(\sinh x)$

(e) $D_x(x^{\sin x})$

$x^{\sin x} = e^{(\ln x)\sin x}$

So

$D_x(x^{\sin x}) = D_x\left(e^{(\ln x)\sin x}\right) = e^{(\ln x)\sin x}\left(\frac{1}{x}\sin x + (\ln x)\cos x\right)$

$= x^{\sin x}\left(\frac{\sin x}{x} + (\ln x)\cos x\right)$

2. (20pts) Evaluate the following indefinite integrals.

4 pts each

Do not take off for missing "+C", but write it in.

Small mistake -1

wrong, but right idea -2

something right -3

totally wrong -4

(a) $\int xe^{x^2}\,dx \quad = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C$

$u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}e^{x^2} + C$

(b) $\int \frac{2}{3x+1}\,dx \quad = \frac{2}{3}\int \frac{du}{u} = \frac{2}{3}\ln|u| + C$

$u = 3x+1$

$du = 3\,dx$

$= \frac{2}{3}\ln|3x+1| + C$

(c) $\int x\ln x\,dx \quad = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$

IBP:

$u = \ln x \quad du = \frac{1}{x}dx$

$dv = x\,dx \quad v = \frac{1}{2}x^2$

$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

(d) $\int \sin(2x)\sin(5x)\,dx \quad = \frac{1}{2}\int (\cos(-3x) - \cos(7x))\,dx$

$= +\frac{1}{6}\sin(3x) - \frac{1}{14}\sin(7x) + C$

or $-\frac{1}{6}\sin(-3x) - \frac{1}{14}\sin(7x) + C$

(e) $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx \quad = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}(u) + C$

$u = e^{2x}$

$du = 2e^{2x}\,dx$

$= \frac{1}{2}\sin^{-1}(e^{2x}) + C$

3. (10pts) The population of a certain species of penguin in the Arctic was found to be 3,000 in 1998 and 4,000 in 2008. Let $P(t)$ denote the penguin population (in thousands) t years since 1998 (that is $P(0) = 3$ and $P(10) = 4$). Assume that the population grows exponentially.

(a) (8pts) Write a formula for $P(t)$. That is, find constants C and k such that $P(t) = Ce^{kt}$.

$$3 = P(0) = Ce^{k \cdot 0} = C. \quad 4$$

$$4 = P(10) = 3e^{10k} \Rightarrow e^{10k} = \frac{4}{3} \Rightarrow k = \frac{1}{10} \ln\left(\frac{4}{3}\right).$$

4

So

$$P(t) = 3e^{\left(\frac{1}{10}\ln\left(\frac{4}{3}\right)\right)t}$$

(b) (2pts) What is the current penguin population? No need to simplify.

$$P(14) = 3e^{\left(\frac{1}{10}\ln(4/3)\right)(14)} \quad 2$$

4. (10pts) Evaluate the following:

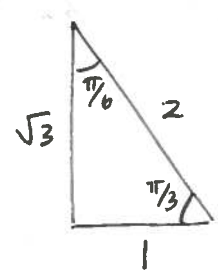
(a) $\sin^{-1}(0) =$ 0 2

(b) $\cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) =$ $\pi/3$ 2

(c) $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) =$ $\pi/4$ 2

(d) $\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right) =$ $\sqrt{3}/2$ 2

(e) $\cos^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right) =$ $\pi/3$ 2



5. (10pts) Use integration by parts to find $\int x^2 e^{2x}\,dx$

$$\int x^2 e^{2x}dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x} dx = \frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx\right)$$

$u = x^2$ $du = 2x\,dx$ $u = x$ $du = dx$

$dv = e^{2x}$ $v = \frac{1}{2}e^{2x}$ $dv = e^{2x}$ $v = \frac{1}{2}e^{2x}$

$$= \boxed{\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C}$$

Correct use of IBP 5

Switching u + dv -4

only one IBP -4

6. (10pts) Find $\int \sin^2 x \cos^2 x\,dx$

$$\int \sin^2 x \cos^2 x\,dx = \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\,dx \quad 5$$

$$= \frac{1}{4}\int (1 - \cos^2(2x))dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2}(1+\cos(4x))\right)dx \quad 2$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos(4x)\right)dx$$

$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C \quad 3$$

7. (10pts) Find the indefinite integral $\int x\sqrt{4-x^2}\,dx$

Set $x = 2\sin t \Rightarrow \sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = 2\cos t$

2 $dx = 2\cos t\,dt$

2

$$\int x\sqrt{4-x^2}\,dx = 8\int \sin t\cos^2 t\,dt = -8\int u^2\,du$$

2

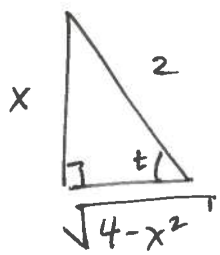
$u = \cos t.$

$du = -\sin t\,dt$

$$= -\frac{8}{3}u^3 + C$$

$$= -\frac{8}{3}(\cos t)^3 + C$$

2



$$= -\frac{1}{3}(4-x^2)^{3/2} + C$$

2

or

$u = 4-x^2$

$du = -2x\,dx$

$$\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int u^{1/2}\,du$$

$$= -\frac{1}{3}u^{3/2} + C$$

$$= -\frac{1}{3}(4-x^2)^{3/2} + C$$

8. (10pts) Find $\int \frac{1}{(x-1)(x+2)}\,dx$

3

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

when $x=1$,

$1 = 3A \Rightarrow A = \frac{1}{3}.$

4

when $x=-2$

$1 = -3B \Rightarrow B = -\frac{1}{3}.$

3

$$\int \frac{1}{(x-1)(x+2)}\,dx = \int \left(\frac{1/3}{x-1} - \frac{1/3}{x+2}\right)dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$$

no absolute values -1

Trigonometric Formulas:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin 2x = 2 \sin x \cos x$$
$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$
$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$
$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

Inverse Trigonometric Formulas:

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1$$
$$D_x \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1$$
$$D_x \tan^{-1} x = \frac{1}{1+x^2}$$
$$D_x \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}} \qquad |x| > 1$$

Hyperbolic Trigonometric Forumlas:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x}$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$D_x \sinh x = \cosh x$$
$$D_x \cosh x = \sinh x$$