Calculus II 1220-90 Final Exam
Summer 2014

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Answers unaccompanied by work may not receive credit. Please try to do all work in the space provided and circle your final answers. The last page contains some useful formulas.

1. (10pts) Compute the following derivatives:

   (a) (5pts) $D_x(\sin(e^{4x}))$

5

$$= \cos(e^{4x}) \cdot e^{4x} \cdot 4 = 4e^{4x}\cos(e^{4x})$$

   (b) (5pts) $D_x(x^{\ln x})$ $= D_x\left(e^{(\ln x)^2}\right)$

5

$$= e^{(\ln x)^2}\left(2\ln x \cdot \frac{1}{x}\right) = x^{\ln x}\left(\frac{2}{x}\ln x\right)$$

2. (10pts) Use L'Hôpital's Rule to compute the following limits. Make sure you indicate where L'Hôpital's Rule is being used.

   (a) (5pts) $\lim_{x\to 0} \frac{\tan^{-1} x}{x}$ $\overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = \lim_{x\to 0} \frac{1}{1+x^2} = 1.$

5

   (b) (5pts) $\lim_{x\to 0} \frac{\cosh x - 1 - \frac{x^2}{2}}{x^3}$ $\overset{L'H}{=} \lim_{x\to 0} \frac{\sinh x - x}{3x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{\cosh x - 1}{6x}$

5

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\sinh x}{6} = 0.$$

3. (5pts) Find a formula for the inverse of the function below. **Hint:** Complete the square.

$$y = f(x) = \sqrt{x^2+6x}$$

5

$$x = \sqrt{y^2+6y}$$

$$x^2 = y^2+6y = (y+3)^2-9 \implies x^2+9 = (y+3)^2$$

$$y = \sqrt{x^2+9} - 3$$

25

4. (24pts) Evaluate the following indefinite integrals. Remember $+C$!

(a) (6pts) $\int x^{-2} \ln x dx$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = x^{-2} dx \Rightarrow v = -x^{-1}$

$= -x^{-1} \ln x + \int \frac{1}{x^2} dx$

$= -\frac{\ln x}{x} - \frac{1}{x} + C$

6

(b) (6pts) $\int \cos^2 x \sin^5 x \, dx$ $= \int \cos^2 x \cdot \sin x \cdot \sin^4 x \, dx$

$= \int \cos^2 x \cdot \sin x \cdot (1 - \cos^2 x)^2 dx = -\int u^2 (1-u^2)^2 du$

$u = \cos x$

$du = -\sin x \, dx$

$= -\int (u^2 - 2u^4 + u^6) du = -\frac{1}{3} u^3 + \frac{2}{5} u^5 - \frac{1}{7} u^7 + C$

$= -\frac{1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x + C$

6

(c) (6pts) $\int \sin(2x) \sin(5x) \, dx$.

$= \int \frac{1}{2} (\cos(-3x) - \cos(7x)) dx = \frac{1}{2} \int (\cos(3x) - \cos(7x)) dx$

$= \frac{1}{6} \sin(3x) - \frac{1}{14} \sin(7x) + C$

6

(d) (6pts) $\int \frac{1}{x^2 - 1} dx$. $= \int \frac{1}{(x+1)(x-1)} dx$

Use partial fractions

$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)} \Rightarrow$ $A + B = 0$, $-A + B = 1$

$A = -\frac{1}{2}$

$B = \frac{1}{2}$.

6

$\Rightarrow \int \frac{1}{x^2-1} dx = -\frac{1}{2} \int \frac{1}{x+1} dx + \frac{1}{2} \int \frac{1}{x-1} dx$

$= -\frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C$

abs. value

24

5. (12pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent. No work is necessary.

C D $\sum_{n=1}^{\infty} \frac{n!}{5^n}$

C D $\sum_{n=1}^{\infty} \frac{n^5}{n^6+1}$

12

C D $\sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^{n-1}$

C D $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^5}}$

C D $\sum_{n=1}^{\infty} \frac{n+2}{n+5}$

C D $\sum_{n=1}^{\infty} (-1)^{n-1} e^{-n}$

6. (12pts) Perform the integral test in two steps:

(a) (6pts) Use **substitution** to find

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx.$$

$u = \sqrt{x} = x^{1/2} \implies du = \frac{1}{2} x^{-1/2} dx = \frac{1}{2\sqrt{x}}\, dx$

6

$= 2\int e^{-u}\, du = -2e^{-u} + C = -2e^{-\sqrt{x}} + C$

(b) (6pts) Use the above and the **integral test** to determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}.$$

$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = \lim_{t\to\infty} \int_1^{t} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = \lim_{t\to\infty} \left(-2e^{-\sqrt{x}}\Big|_1^t\right.$

6

$= \lim_{t\to\infty} \left(-2e^{-\sqrt{t}} + 2e^{-1}\right)$

$= 2e^{-1} < \infty.$

By integral test,

$\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converges.

24

7. (9pts) Find the **interval** of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x+2)^n}{3^{n+1}}.$$

Use absolute Ratio Test: $a_n = \frac{|x+2|^n}{3^{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x+2|^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{|x+2|^n} = \lim_{n\to\infty} \frac{|x+2|}{3} < 1 \Rightarrow$$

9

converges absolutely on $|x+2| < 3$ or $(-5, 1)$

check endpoints:

$x=-5 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-3)^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{3}$ diverges

$x=1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}3^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3}$ diverges

So interval of conv.

$(-5, 1)$

8. (14pts) Let

$$f(x) = \sqrt{1+x^2} = (1+x^2)^{1/2}$$

(a) (6pts) Find the following derivatives of $f(x)$ evaluated at $x=0$:

6

$f(0) =$ 1

$f'(0) =$ 0

$f''(0) =$ 1

$f(0) = (1)^{1/2} = 1$

$f'(x) = \frac{1}{2}(1+x^2)^{-1/2}(2x) = \frac{x}{\sqrt{1+x^2}}$

$f'(0) = \frac{0}{1} = 0.$

$f''(x) = (1+x^2)^{-1/2} + x(-\frac{1}{2})(1+x^2)^{-3/2}(2x)$

$f''(0) = 1.$

(b) (4pts) Use your computations above to write out $P_2(x)$, the Maclaurin polynomial of degree 2 for $f(x)$.

4

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 + \frac{1}{2}x^2$$

(c) (4pts) Now use $P_2(x)$ to estimate

$$\int_0^{1/2} \sqrt{1+x^2}\, dx$$

4

$$\approx \int_0^{1/2} (1+\tfrac{1}{2}x^2)\,dx = \left(x + \tfrac{1}{6}x^3\right)\Big|_0^{1/2} = \frac{1}{2} + \frac{1}{6}\cdot\frac{1}{8}$$

$$= \frac{1}{2} + \frac{1}{48} = \frac{25}{48}.$$

23

9. (8pts) Consider the conic section

$$2x^2 - 4x + 2y^2 + 20y = -44.$$

(a) (2pts) What type of conic section is this? Circle one answer below:

Parabola   Ellipse   Hyperbola   Circle

(b) (4pts) The center of the conic section is the point (1 , -5).

8

(c) (2pts) The shortest distance from the center to a point on the conic section is 2.

$$2x^2-4x+2y^2+20y=-44$$
$$2(x^2-2x)+2(y^2+10y)=-44$$
$$(x^2-2x)+(y^2+10y)=-22$$
$$(x-1)^2-1+(y+5)^2-25=-22$$
$$(x-1)^2+(y+5)^2=4=2^2$$

10. (6pts) Match the Cartesian coordinates (x, y) on the left to the corresponding polar coordinates (r, θ) on the right by writing the letter in the blank provided. Every answer will be used exactly once.

6

B $(0, 2)$ A. $(2, -\frac{\pi}{4})$

C $(-\sqrt{3}, -1)$ B. $(2, \frac{\pi}{2})$

A $(\sqrt{2}, -\sqrt{2})$ C. $(2, \frac{7\pi}{6})$

11. (6pts) Match the polar coordinates (r, θ) on the left to the corresponding Cartesian coordinates (x, y) on the right by writing the letter in the blank provided. Every answer will be used exactly once.

6

C $(2, 2\pi)$ A. $(-\sqrt{2}, \sqrt{2})$

A $(2, \frac{3\pi}{4})$ B. $(1, \sqrt{3})$

B $(2, \frac{\pi}{3})$ C. $(2, 0)$

12. (14pts) Match the polar equation to the type of curve it determines by writing the letter in the blank provided. Every answer will be used exactly once.

14

G $r = \tan\theta \sec\theta$ A. a circle centered at the origin

A $r = 6$ B. a circle centered on the y-axis

B $r = 4\sin\theta$ C. a vertical line

F $r = \sqrt{\dfrac{1}{\cos^2\theta + 2\sin^2\theta}}$ D. an angled line

E $r = 12\cos\theta$ E. a circle centered on the x-axis

D $r = \dfrac{1}{\sin\theta - \cos\theta}$ F. an ellipse

C $r = \sec\theta$ G. a parabola

34

13. (20pts) Consider the curve determined by the polar equation $r = \cos(2\theta)$.

(a) (4pts) At what angles θ with $0 \le \theta < 2\pi$ is r equal to zero?

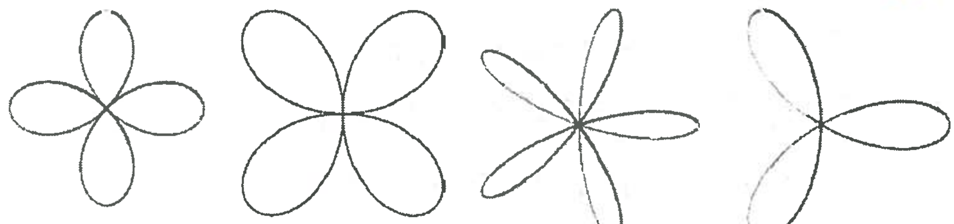
4

$$0 = \cos(2\theta)$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

(b) (4pts) Consequently, which of the following is a graph of the curve $r = \cos(2\theta)$ A

4



A B C D

(c) (8pts) Find the area of one loop.

$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1}{2}(1+\cos(4\theta))\, d\theta$$

8

$$= \frac{1}{4}\left(\theta + \frac{1}{4}\sin(4\theta)\right)\Big|_{-\pi/4}^{\pi/4} = \frac{1}{4}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi}{8}$$

(d) (4pts) Set up an integral which gives the total length of the curve. You **do not** have to evaluate it.

$$f(\theta) = \cos(2\theta)$$

$$f'(\theta) = -2\sin(2\theta)$$

4

$$L = \int_0^{2\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\cos^2(2\theta) + 4\sin^2(2\theta)}\, d\theta.$$

20