1220-90 Final Exam

Summer 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Answers unaccompanied by work may not receive credit. Please try to do all work in the space provided and circle your final answers. The last page contains some useful formulas.

1. (10pts) Compute the following derivatives:

   (a) (5pts) $D_x(\tan^{-1}(x^2))$

   $$= \underbrace{\frac{1}{1+(x^2)^2}}_{3} \cdot \underbrace{2x}_{2} = \frac{2x}{1+x^4}$$

   5

   (b) (5pts) $D_x(x^{7x})$

   $$x^{7x} = e^{7x \ln x} \quad 2$$

   5

   $$D_x\left(e^{7x\ln x}\right) = e^{7x \ln x}\left(7\ln x + 7x\left(\tfrac{1}{x}\right)\right) = x^{7x}(7\ln x + 7)$$

2. (15pts) Compute the following indefinite integrals: **Remember:** $+C$!

   Missing $+C$: -2 total

   (a) (5pts) $\int \frac{1}{4x+3}\,dx$ $= \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\ln|u| + C$

   5

   $u = 4x+3$

   $du = 4\,dx$

   $$= \frac{1}{4}\ln|4x+3| + C$$

   (b) (5pts) $\int \cos^3 x\,dx$ $= \int \cos^2 x(\cos x)\,dx = \int (1-\sin^2 x)\cos x\,dx$

   $u = \sin x$

   $du = \cos x\,dx$

   5

   $$= \int (1-u^2)\,du$$

   $$= u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C$$

   (c) (5pts) $\int x^2 \ln x\,dx$ $= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$

   $u = \ln x \qquad du = \frac{1}{x}\,dx$

   $dv = x^2\,dx \qquad v = \frac{x^3}{3}$

   5

   $$= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$$

25

3. (5pts) a formula for the inverse of the function

$$f(x) = \ln(3x+8)$$

$x = \ln(3y+8)$

5

$e^x = e^{\ln(3y+8)} = 3y+8 \implies \boxed{y = \frac{1}{3}(e^x - 8)}$

4. (10pts) Use the rationalizing substitution $x = \tan t$ to find the antiderivative

$$\int \frac{1}{(x^2+1)^{3/2}}\, dx$$

$x = \tan t \Rightarrow dx = \sec^2 t\, dt$

$x^2 + 1 = \tan^2 t + 1 = \sec^2 t$

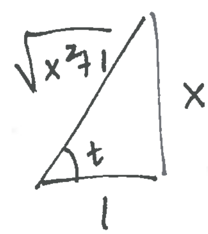
5

So

$$\int \frac{1}{(x^2+1)^{3/2}}\, dx = \int \frac{1}{(\sec^2 t)^{3/2}} \sec^2 t\, dt = \int \frac{1}{\sec t}\, dt$$

$$= \int \cos t\, dt$$

10



$$= \sin t + C.$$

8

$$= \frac{x}{\sqrt{x^2+1}} + C$$

5. (10pts) Find the antiderivative $\int \frac{x}{x^2 - 3x + 2}\, dx$.

$$\frac{x}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{A(x-1) + B(x-2)}{x^2-3x+2}$$

$$= \frac{(A+B)x + (-A-2B)}{x^2-3x+2}$$

10

So $A + B = 1$

$-A - 2B = 0$

$-B = 1 \implies B = -1$

$A = 2$

So

$$\int \frac{x}{x^2-3x+2}\, dx = \int \frac{2}{x-2}\, dx + \int \frac{-1}{x-1}\, dx$$

$$= \boxed{2 \ln|x-2| - \ln|x-1| + C}$$

25

6. (14pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent. No work is necessary.

14

C D $\sum_{n=1}^{\infty} \frac{1}{\ln n}$

C D $\sum_{n=1}^{\infty} \frac{4n}{3n+9}$

C D $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

C D $\sum_{n=1}^{\infty} e^{-n}$

C D $\sum_{n=1}^{\infty} \frac{6^n}{n!}$

C D $\sum_{n=1}^{\infty} \left(\frac{7}{6}\right)^{n-1}$

C D $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^2+1}}$

7. (10pts) Use the integral test to determine whether the series

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$$

converges or diverges.

10

$$\int_3^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_3^t \frac{1}{x(\ln x)^2}\,dx$$

$u = \ln x$

$du = \frac{1}{x}dx$

$$= \lim_{t\to\infty} \int_{\ln 3}^{\ln t} u^{-2}\,du$$

$$= \lim_{t\to\infty} \left(-u^{-1}\Big|_{\ln 3}^{\ln t}\right.$$

$$= \lim_{t\to\infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln 3}\right) = \frac{1}{\ln 3} < \infty.$$

So $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ converges

8. (10pts) Find the **interval** of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n}(-5)^n}$$

Use Ratio Test to test for abs. conv. $a_n = \frac{|x-2|^n}{\sqrt{n}\,(+5)^n}$

$$\frac{a_{n+1}}{a_n} = \frac{|x-2|^{n+1}}{\sqrt{n+1}\,5^{n+1}} \cdot \frac{5^n\sqrt{n}}{|x-2|^n} = \frac{|x-2|}{5}\sqrt{\frac{n}{n+1}}.$$

10

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x-2|}{5}\sqrt{\frac{n}{n+1}} = \frac{|x-2|}{5} < 1$$

So series converges absolutely on $(-3,7)$

Test endpoints:

$x=-3$, $\sum_{n=1}^{\infty} \frac{(-3-2)^n}{\sqrt{n}(-5)^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

$x=7$ $\sum_{n=1}^{\infty} \frac{(7-2)^n}{\sqrt{n}(-5)^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges

interval of convergence $(-3,7]$

9. (12pts) Let

$$f(x) = \ln(1+x+x^2)$$

(a) (6pts) Find the following derivatives of $f(x)$ evaluated at $x=0$:

6

$f(0) =$ $\ln(1) = 0$

$f'(0) =$ 1

$f''(0) =$ 1

$$f'(x) = \frac{1}{1+x+x^2}(2x+1) \Rightarrow f'(0) = \frac{1}{1} = 1$$

$$f''(x) = \frac{(1+x+x^2)(2) - (2x+1)(2x+1)}{(1+x+x^2)^2}$$

$$\Rightarrow f''(0) = 1$$

(b) (3pts) Use your computations above to write out $P_2(x)$, the second degree MacLaurin polynomial for $f(x)$.

3

$$P_2(x) = 0 + x + \tfrac{1}{2}x^2$$

(c) (3pts) Use $P_2(x)$ to evaluate the limit

3

$$\lim_{x\to 0} \frac{\ln(1+x+x^2) - x}{x^2}.$$

+ higher order

$$= \lim_{x\to 0} \frac{(x+\frac{1}{2}x^2) - x}{x^2} = \lim_{x\to 0} \frac{\frac{1}{2}x^2 + \text{higher order}}{x^2}$$

$$= \frac{1}{2}$$

10. (12pts) Match the equation with the type of conic section it describes by writing the letter in the blank provided.

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C	$x + y^2 - 2y = 1$	A. circle
E	$x^2 + y^2 - 2y = -1$	B. ellipse
F	$x^2 + y^2 - 2y = -3$	C. parabola
A	$x^2 + y^2 - 2y = 0$	D. hyperbola
B	$2x^2 + y^2 - 2y = 1$	E. a point
D	$x^2 - y^2 - 2y = 0$	F. the empty set (no solution)

11. (a) (4pts) Find the polar coordinates of the point with Cartesian coordinates $(\sqrt{2}, \sqrt{2})$.

$r^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 4 \Rightarrow r = 2.$ $r^2 = 4$ -1

4

$\theta = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$

$(r, \theta) = (2, \frac{\pi}{4})$

(b) (4pts) Find the Cartesian coordinates of the point with polar coordinates $(4, \frac{2\pi}{3})$.

$x = r\cos\theta = 4\cos\left(\frac{2\pi}{3}\right) = 4\left(-\frac{1}{2}\right) = -2$

4

$y = r\sin\theta = 4\sin\left(\frac{2\pi}{3}\right) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$

$(x, y) = (-2, 2\sqrt{3})$

12. (14pts) Match the polar equation to the type of curve it determines by writing the letter in the blank provided. Every answer will be used exactly once.

14

C	$r = \frac{6}{\sin\theta}$	A. a circle centered at the origin
G	$r = e^{-\theta}$	B. a circle centered on the y-axis
E	$r = \sec\theta$	C. a horizontal line
B	$r = 8\sin\theta$	D. an angled line
A	$r = \pi$	E. a vertical line
D	$r = \frac{1}{\sin\theta - \cos\theta}$	F. a circle centered on the x-axis
F	$r = 4\cos\theta$	G. a spiral

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13. (20pts) Consider the limaçon determined by the polar equation $r = 2 + \sin\theta$. A graph of this polar curve is given a the bottom of the page.

(a) (6pts) Find the slope of the tangent line to the curve at the point $(2, 0)$ (that is, when $\theta = 0$).

$$f(\theta) = 2 + \sin\theta \Rightarrow f(0) = 2 \quad (1)$$

$$f'(\theta) = \cos\theta \Rightarrow f'(0) = 1 \quad (1)$$

6

$$m = \frac{f'(0)\sin(0) + f(0)\cos(0)}{f'(0)\cos(0) - f(0)\sin(0)} \ (2) = \frac{1\cdot 0 + 2\cdot 1}{1\cdot 1 - 2\cdot 0} \ (1) = \frac{2}{1} \ (1) = 2$$

(b) (9pts) Find the area of the region inside the limaçon.

$$A = \frac{1}{2}\int_0^{2\pi} (2+\sin\theta)^2\, d\theta \ (2) = \frac{1}{2}\int_0^{2\pi} (4 + 4\sin\theta + \sin^2\theta)\, d\theta \ (2)$$

9

$$= 4\pi + \frac{1}{2}\int_0^{2\pi} \sin^2\theta\, d\theta \ (1)$$

$$= 4\pi + \frac{1}{4}\int_0^{2\pi} (1 - \cos(2\theta))\, d\theta \ (2)$$

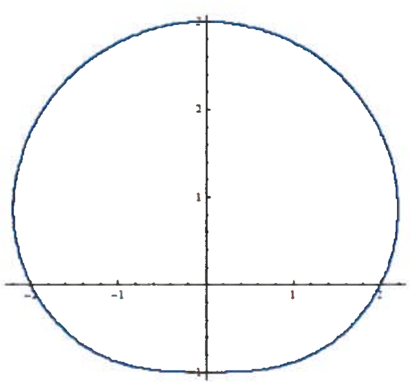
$$= 4\pi + \frac{\pi}{2} = \frac{9\pi}{2} \ (2)$$

(c) (5pts) Set up an integral to find the perimeter of the limaçon. Do not attempt to solve.

$$L = \int_0^{2\pi} \sqrt{(2+\sin\theta)^2 + \cos^2\theta}\, d\theta \quad (1)\ (1)\ (1)$$

5

$$(2) \quad = \int_0^{2\pi} \sqrt{5 + 4\sin\theta}\, d\theta.$$



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