1220-90 Final Exam
Spring 2014

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Answers unaccompanied by work may not receive credit. Please try to do all work in the space provided and circle your final answers. The last page contains some useful formulas.

1. (15pts) Compute the following derivatives:

   (a) (5pts) $D_x(\sin^{-1}(x^4))$ $= \frac{1}{\sqrt{1-(x^4)^2}}(4x^3) = \frac{4x^3}{\sqrt{1-x^8}}$

   5

   (b) (5pts) $D_x(\ln(\cos x))$

   $= \frac{1}{\cos x}(-\sin x) = -\frac{\sin x}{\cos x} = -\tan x$

   5

   (c) (5pts) $D_x(x^{4x})$

   5

   $x^{4x} = e^{4x\ln x}$

   $D_x\left(e^{4x\ln x}\right) = e^{4x\ln x}\left(4\ln x + 4x\left(\frac{1}{x}\right)\right) = x^{4x}(4\ln x + 4)$

2. (15pts) Compute the following indefinite integrals. **Remember:** $+C$!

   -1

   (a) (5pts) $\int \frac{1}{3x-2}\,dx$ $= \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C$

   5

   $u = 3x-2$

   $du = 3\,dx$

   $= \frac{1}{3}\ln|3x-2| + C$

   (b) (5pts) $\int \tan^2 x \sec^2 x\,dx$ $= \int u^2\,du = \frac{u^3}{3} + C$

   5

   $u = \tan x$

   $du = \sec^2 x\,dx$

   $= \frac{\tan^3 x}{3} + C$

   (c) (5pts) $\int \cos^2 x\,dx$

   5

   $= \frac{1}{2}\int (1+\cos(2x))\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C$

   $= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

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3. (10pts) Use integration by parts to compute the following indefinite integrals: **Remember:** $+C$!

(a) (5pts) $\int x \sin(3x)\, dx \quad = -\frac{x}{3}\cos(3x) + \frac{1}{3}\int \cos(3x)\,dx$

$u = x \Rightarrow du = dx$

$dv = \sin(3x)dx \Rightarrow v = -\frac{1}{3}\cos(3x) \quad = -\frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x) + C$

5

(b) (5pts) $\int x^3 \ln x\, dx \quad = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx$

$u = \ln x \Rightarrow du = \frac{1}{x}dx$

$dv = x^3 dx \Rightarrow v = \frac{x^4}{4} \quad = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$

5

4. (10pts) Evaluate $\int_3^5 \frac{-x+8}{x^2-x-2}\, dx$

$\frac{-x+8}{x^2-x-2} = \frac{-x+8}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$

plug in $x=-1$: $9 = -3B \Rightarrow B=-3 \Longrightarrow -x+8 = A(x+1) + B(x-2)$

plug in $x=2$: $6 = 3A \Rightarrow A=2$ 4

10

$\int_3^5 \frac{-x+8}{x^2-x-2}dx = \int_3^5 \left(\frac{2}{x-2} - \frac{3}{x+1}\right)dx$

$= \left(2\ln|x-2| - 3\ln|x+1|\right)\Big|_3^5$

$= (2\ln 3 - 3\ln 6) - (\cancel{2\ln 1}^{\,0} - 3\ln 4)$

$= 2\ln 3 - 3\ln 6 + 3\ln 4$

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5. (12pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent. No work is necessary.

C D $\sum_{n=1}^{\infty} \frac{n^2+1}{3n}$

C D $\sum_{n=1}^{\infty} \frac{n}{4^n}$

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C D $\sum_{n=1}^{\infty} 5\left(\frac{3}{4}\right)^{n-1}$

C D $\sum_{n=1}^{\infty} (-1)^{n-1}e^n$

C D $\sum_{n=1}^{\infty} \cos(n\pi)$

C D $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

6. (12pts) Perform the integral test in two steps:

(a) (6pts) Use **substitution** to find

$$\int \frac{2x+1}{(x^2+x)^5}\,dx.$$

3

$u = x^2 + x \Rightarrow du = (2x+1)\,dx$

-1 missing $+C$

6 $= \int u^{-5}\,du = -\frac{1}{4}u^{-4} + C = -\frac{1}{4}(x^2+x)^{-4} + C$

(b) (6pts) Use the above and the **integral test** to determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^5}.$$

6

$$\int_1^{\infty} \frac{2x+1}{(x^2+x)^5}\,dx = \lim_{t\to\infty} \int_1^t \frac{2x+1}{(x^2+x)^5}\,dx = \lim_{t\to\infty}\left(-\frac{1}{4}(x^2+x)^{-4}\Big|_1^t\right)$$

$$= \lim_{t\to\infty}\left(\frac{-1}{4(t^2+t)^4} + \frac{1}{4(1^2+1)^4}\right) = \frac{1}{64}.$$

So since $\int_1^{\infty} \frac{2x+1}{(x^2+x)^5}\,dx < \infty$, by the integral test,

$\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^5}$ converges

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7. (14pts) Let

$$f(x) = e^{2x}$$

(a) (3pts) Find the following derivatives of $f(x)$ evaluated at $x = 1$:

3

$f(1) =$ e^2

$f'(1) =$ $2e^2$

$f''(1) =$ $4e^2$

$f(x) = e^{2x} \Rightarrow f(1) = e^2$
$f'(x) = 2e^{2x} \Rightarrow f'(1) = 2e^2$
$f''(x) = 4e^{2x} \Rightarrow f''(1) = 4e^2$

(b) (3pts) Find a formula (seeing the pattern in your computations above) for the nth derivative of $f(x)$ at $x = 1$:

3

$f^{(n)}(1) =$ $2^n e^2$

(c) (2pts) Write out the full Taylor series for $f(x)$ at $x = 1$ using the formula you found for $f^{(n)}(1)$ above:

2

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(x-1)^n$$

(d) (6pts) Use the Ratio Test to determine the interval of convergence of the Taylor series you found in part (c) above.

6

$$a_n = \frac{2^n e^2}{n!}|x-1|^n$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} e^2 |x-1|^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n e^2 |x-1|^n} = \frac{2|x-1|}{n+1} \longrightarrow 0 \text{ as } n \to \infty$$

for any x.

So interval of convergence is $(-\infty, \infty)$.

8. (6pts) Use the fact that $\cos x$ is equal to its Maclaurin series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots$$

to find the 4th derivative of $f(x) = \cos(3x^2)$ at $x = 0$.

6

$$\cos(3x^2) = 1 - \frac{(3x^2)^2}{2} + \frac{(3x^2)^4}{24} - \frac{(3x^2)^6}{720} + \ldots$$

$$= 1 - \frac{9x^4}{2} + \frac{81x^8}{24} - \ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

So $$\frac{-9}{2} = \frac{f^{(4)}(0)}{4!} = \frac{f^{(4)}(0)}{24} \Rightarrow f^{(4)}(0) = 24\left(\frac{-9}{2}\right) = -108$$

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9. (12pts) Match the equation with the type of conic section it describes by writing the letter in the blank provided.

- F $x^2 + 4x + y^2 - 2y = -6$
- E $x^2 + 4x + y^2 - 2y = -5$
- A $x^2 + 4x + y^2 - 2y = -4$
- D $x^2 + 4x - y^2 + 2y = -2$
- B $2x^2 + 8x + y^2 - 2y = -9$
- C $4x + y^2 - 2y = -4$

A. circle
B. ellipse
C. parabola
D. hyperbola
E. a point
F. the empty set (no solution)

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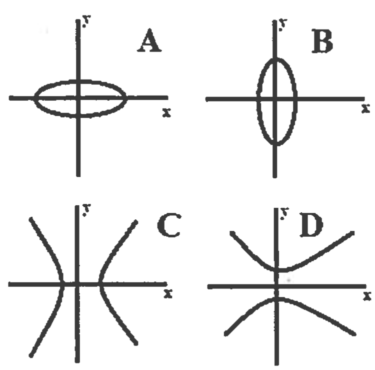
10. (8pts) Match the equation of the conic section to its graph by writing the letter in the blank provided. Every answer will be used exactly once.

- C $\frac{x^2}{2^2} - \frac{y^2}{3^3} = 1$
- B $\frac{x^2}{2^2} + \frac{y^2}{3^3} = 1$
- A $\frac{x^2}{3^2} + \frac{y^2}{2^3} = 1$
- D $\frac{y^2}{2^2} - \frac{x^2}{3^3} = 1$



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11. (14pts) Match the polar equation to the type of curve it determines by writing the letter in the blank provided. Every answer will be used exactly once.

- G $r = \theta$
- E $r = \dfrac{5}{\cos\theta}$
- D $r = \dfrac{2}{\sin\theta + \cos\theta}$
- C $r = \dfrac{-2}{\sin\theta}$
- A $r = 1$
- B $r = 4\sin\theta$
- F $r = 2\cos\theta$

A. a circle centered at the origin
B. a circle centered on the y-axis
C. a horizontal line
D. an angled line
E. a vertical line
F. a circle centered on the x-axis
G. a spiral

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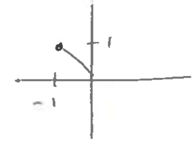
12. (8pts) Convert between polar and Cartesian Coordinates as indicated.

(a) (4pts) Find the polar coordinates of the point with Cartesian coordinates $(-1, 1)$.

4

$r = \sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\theta = \frac{y}{x} = -1 \implies \theta = \frac{3\pi}{4}$



$\left(\sqrt{2}, \frac{3\pi}{4}\right)$ since in 2^{nd} quadrant.

(b) (4pts) Find the Cartesian coordinates of the point with polar coordinates $(3, \frac{3\pi}{2})$.

4

$x = r\cos\theta = 3\cos\left(\frac{3\pi}{2}\right) = 0$

$y = r\sin\theta = 3\sin\left(\frac{3\pi}{2}\right) = -3.$

$(0, -3)$

13. (14pts) Consider the curve determined by the polar equation $r = \theta^2$. A graph of this polar curve, between $\theta = 0$ and $\theta = \pi$ is given a the bottom of the page.

(a) (7pts) Find the area of the region inside the curve and above the x-axis (between $\theta = 0$ and $\theta = \pi$).

$f(\theta) = \theta^2$

7

$$A = \int_0^\pi \frac{1}{2}(\theta^2)^2\,d\theta = \frac{1}{2}\int_0^\pi \theta^4\,d\theta = \frac{1}{2}\left(\frac{\theta^5}{5}\Big|_0^\pi\right. = \frac{\pi^5}{10}$$

(b) (7pts) Find the length of the curve between $\theta = 0$ and $\theta = \pi$.

$f(\theta) = \theta^2 \Rightarrow f'(\theta) = 2\theta.$

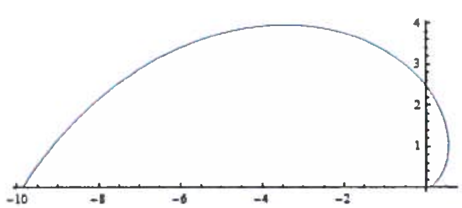
7

$$L = \int_0^\pi \sqrt{\theta^4 + 4\theta^2}\,d\theta = \int_0^\pi \theta\sqrt{\theta^2+4}\,d\theta = \frac{1}{2}\int_4^{\pi^2+4} u^{1/2}\,du$$

$u = \theta^2+4$

$du = 2\theta\,d\theta$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\Big|_4^{\pi^2+4}\right.$$



$$= \frac{1}{3}\left[(\pi^2+4)^{3/2} - 8\right]$$

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