1220-90 Final Exam

Fall 2012

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Please try to do all all work in the space provided. Please circle your final answer. The last page contains some useful identities.

1. (15pts) Compute the following ~~derivatives~~:

   (a) (5pts) $D_x(\log_5 x)$ $= \frac{1}{x \ln 5}$

   (b) (5pts) $D_x(3^{x^2})$ $= 3^{x^2}(\ln 3)(2x)$

   (c) (5pts) $D_x((2x)^{3x})$

   $D_x((2x)^{3x}) = D_x((e^{\ln 2x})^{3x}) = D_x(e^{3x\ln(2x)}) =$

   $e^{3x\ln(2x)}(3\ln(2x)+3) = (2x)^{3x}(3\ln(2x)+3)$

2. (15pts) Compute the following indefinite integrals: **Remember:** $+C$!

   (a) (5pts) $\int \frac{1}{\sqrt{1-x^2}}\,dx$ $= \sin^{-1}x + C$

   (b) (5pts) $\int x^2\sqrt{x^3+4}\,dx$ $= \frac{1}{3}\int u^{1/2}\,du = \frac{2}{9}(x^3+4)^{3/2} + C$

   $u = x^3+4$

   $du = 3x^2dx$

   (c) (5pts) $\int x\cos x\,dx$ $= x\sin x - \int \sin x\,dx$

   $u = x \quad du = dx$

   $dv = \cos x \quad v = \sin x$

   $= x\sin x + \cos x + C$

3. (12pts) Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ by making the rationalizing substitution $x = 2\tan t$.

$x = 2\tan t \Rightarrow dx = 2\sec^2 t \, dt$

$\sqrt{x^2+4} = \sqrt{4\tan^2 t + 4} = 2\sec t$

$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2 t \, dt}{4\tan^2 t \cdot 2\sec t}$ 4

$= \frac{1}{4}\int \frac{\sec t}{\tan^2 t}\,dt$

$= \frac{1}{4}\int \frac{\cos t}{\sin^2 t}\,dt$ 2

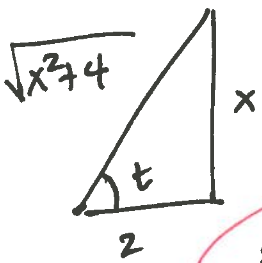
$u = \sin t \Rightarrow du = \cos t\,dt$

$= \frac{1}{4}\int \frac{1}{u^2}\,du$ 2

$= -\frac{1}{4}u^{-1} + C$

$= -\frac{1}{4}\frac{1}{\sin t} + C$ 2

$= -\frac{\sqrt{x^2+4}}{4x} + C$



So $\sin t = \frac{x}{\sqrt{x^2+4}}$ 2

4. (10pts) Use partial fractions to find the antiderivative $\int \frac{2}{x^2-2x}\,dx$.

$\frac{2}{x^2-2x} = \frac{2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$ 4 $= \frac{A(x-2)+Bx}{x(x-2)}$

$A = -1$
$B = 1$ 2

$\int \frac{2}{x^2-2x}\,dx = \int \left(-\frac{1}{x} + \frac{1}{x-2}\right)dx$

$= -\ln|x| + \ln|x-2| + C$

4

5. (12pts) Determine, by whatever method you wish, whether the following series are convergent or divergent. Circle 'C' if the series is convergent or 'D' if the series is divergent. No work is necessary.

C D $\sum_{n=1}^{\infty}(-1)^{n-1}e^{-n}$

C D $\sum_{n=1}^{\infty}\frac{1}{1+n+n^2}$

C D $\sum_{n=1}^{\infty}\frac{2^n}{n!}$

C D $\sum_{n=1}^{\infty}(\frac{1}{n+2}-\frac{1}{n+3})$

C D $\sum_{n=1}^{\infty}5(\frac{7}{6})^{n-1}$

C D $\sum_{n=1}^{\infty}(\frac{n^2}{n^2+n})^4$

# missed	score.
0	12
1	10
2	8
3	6
4	4
5	2
6	0

6. (12pts) Use **either** the integral test or the comparison test to determine whether the series

$$\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{1+n^2}$$

converges or diverges. **Note:** You can assume that this series satisfies the hypotheses of both tests; you do not need to check this.

Integral Test:

$$\int_1^{\infty}\frac{\tan^{-1}x}{1+x^2}dx = \lim_{t\to\infty}\int_1^t\frac{\tan^{-1}x}{1+x^2}dx$$

$u = \tan^{-1}x$

$du = \frac{1}{1+x^2}dx$

$$= \lim_{t\to\infty}\int_{\tan^{-1}(1)}^{\tan^{-1}(t)} u\,du$$

$$= \lim_{t\to\infty}\left(\frac{u^2}{2}\Big|_{\pi/4}^{\tan^{-1}(t)}\right)$$

$$= \lim_{t\to\infty}\left(\frac{\tan^{-1}(t)^2}{2} - \frac{\pi^2}{32}\right)$$

$$= \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32} < \infty$$

$\therefore$ convergent

Comparison Test:

Note that $0 \le |\tan^{-1}n| < \frac{\pi}{2}$ for $n \ge 0$.

So for $n \ge 1$,

$$\frac{\tan^{-1}n}{1+n^2} \le \frac{\pi/2}{1+n^2} \le \frac{\pi/2}{n^2}.$$

Now

$$\sum_{n=1}^{\infty}\frac{\pi/2}{n^2} = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

converges (p-series w/ $p = 2 > 1$)

So by comparison test,

$\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{1+n^2}$ is convergent

7. (10pts) Use the fact that $D_x(\frac{1}{1-x}) = \frac{1}{(1-x)^2}$ and the power series $\frac{1}{1-x} = 1+x+x^2+x^3+... = \sum_{n=1}^{\infty} x^{n-1}$ for $|x| < 1$ to find the radius of convergence and the first few coefficients of the power series

$$\frac{4}{(2+x)^2} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ... = \sum_{n=0}^{\infty} c_n x^n.$$

$c_0 =$ 1
$c_1 =$ −1
$c_2 =$ 3/4
$c_3 =$ −1/2

What is the radius of convergence of this power series? 2

when $|x|<1$

$$\frac{1}{(1-x)^2} = D_x\left(\frac{1}{1-x}\right) = D_x\left(1+x+x^2+x^3+\cdots\right) = 1+2x+3x^2+4x^3+\cdots$$

Now

$$\frac{4}{(2+x)^2} = \frac{4}{4(1+\frac{x}{2})^2} = \frac{1}{(1-(-\frac{x}{2}))^2} = 1+2\left(-\frac{x}{2}\right)+3\left(-\frac{x}{2}\right)^2+4\left(-\frac{x}{2}\right)^3+\cdots$$

$$= 1 - x + \tfrac{3}{4}x^2 - \tfrac{1}{2}x^3 + \cdots$$

when $|\frac{x}{2}|<1 \Rightarrow |x|<2$.

8. (10pts) Find the **interval** of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{3n^2}.$$

Use Ratio test: w/ $a_n = \frac{|x-1|^n}{3n^2}$

$$\frac{a_{n+1}}{a_n} = \frac{|x-1|^{n+1}}{3(n+1)^2} \cdot \frac{3n^2}{|x-1|^n} = |x-1| \frac{n^2}{(n+1)^2}.$$

6

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} |x-1| \frac{n^2}{(n+1)^2} = |x-1| < 1 \Longrightarrow$$ series converges abs. when $|x-1|<1$.

Check endpoints:

x = 0: series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n^2}$, converges by AST

x = 2: series becomes $\sum_{n=1}^{\infty} \frac{1}{3n^2}$, converges by p-series.

So interval of convergence is $[0,2]$

9. (8pts) Consider the ellipse determined by the equation

$$x^2 + 4y^2 - 6x + 8y = 0.$$

(a) (6pts) The center of the ellipse is the point ($\underline{3}$, $\underline{-1}$).

(b) (2pts) The length of the major diameter if the ellipse is $\underline{2\sqrt{13}}$.

$$x^2+4y^2-6x+8y=0$$

$$(x^2-6x+9)-9+4(y^2+2y+1)-4=0$$

$$(x-3)^2+4(y+1)^2=13$$

$$\frac{(x-3)^2}{(\sqrt{13})^2}+\frac{(y+1)^2}{(\sqrt{13}/2)^2}=1.$$

10. (a) (4pts) Find the polar coordinates of the point with Cartesian coordinates $(1, \sqrt{3})$.

$$r^2 = 1^2+(\sqrt{3})^2 = 4 \rightarrow r = 2.$$

$$\tan\theta = y/x = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

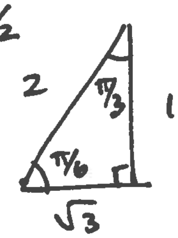
$$(2, \pi/3)$$

(b) (4pts) Find the Cartesian coordinates of the point with polar coordinates $(3, \frac{\pi}{6})$.

$$x = r\cos\theta = 3\cos(\pi/6) = 3\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}/2$$

$$y = r\sin\theta = 3\sin(\pi/6) = 3\left(\tfrac{1}{2}\right) = 3/2$$

$$(3\sqrt{3}/2, 3/2)$$



11. (14pts) Match the polar equation to the type of curve it determines by writing the letter in the blank provided. Every answer will be used exactly once.

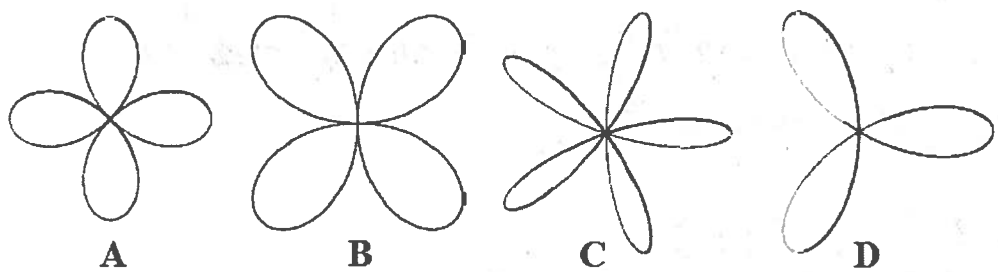
G	$r = \theta$	A. a circle centered at the origin
A	$r = 1$	B. a circle centered on the y-axis
E	$r = 5 + 5\sin\theta$	C. a horizontal line
C	$r = \dfrac{2}{\sin\theta}$	D. an angled line
B	$r = 12\sin\theta$	E. a cardioid
D	$r = \dfrac{1}{2\sin\theta + 5\cos\theta}$	F. a lemniscate
F	$r^2 = 3\cos(2\theta)$	G. a spiral

12. (24pts) Consider the curve determined by the polar equation $r = 3\sin(2\theta)$.

(a) (5pts) At what angles θ with $0 \le \theta < 2\pi$ is r equal to zero?

$$\sin(2\theta) = 0 \Rightarrow \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

(b) (5pts) Consequently, which of the following is a graph of the curve $r = 3\sin(2\theta)$? B



A B C D

(c) (8pts) Find the area of one loop.

$$A = \frac{1}{2}\int_0^{\pi/2} (3\sin(2\theta))^2\,d\theta = \frac{9}{2}\int_0^{\pi/2} \sin^2(2\theta)\,d\theta$$

$$= \frac{9}{4}\int_0^{\pi/2} (1-\cos(4\theta))\,d\theta$$

$$= \frac{9}{4}\left(\theta - \frac{1}{4}\sin(4\theta)\right|_0^{\pi/2}$$

$$= \frac{9}{4}\left(\frac{\pi}{2}\right) = \frac{9\pi}{8}$$

(d) (6pts) Set up an integral which gives the total length of the curve. You **do not** have to evaluate it.

$$f(\theta) = 3\sin(2\theta)$$
$$f'(\theta) = 6\cos(2\theta)$$

$$L = \int_0^{2\pi} \sqrt{9\sin^2(2\theta) + 36\cos^2(2\theta)}\,d\theta.$$

Trigonometric Formulas:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin 2x = 2 \sin x \cos x$$
$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$
$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$
$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

Inverse Trigonometric Formulas:

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1$$
$$D_x \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1$$
$$D_x \tan^{-1} x = \frac{1}{1+x^2}$$
$$D_x \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}} \qquad |x| > 1$$

Calculus with Polar Curve $r = f(\theta)$:

$$A = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2 \, d\theta$$
$$L = \int_a^b \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2} \, d\theta$$
$$m = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$