Calculus I
Practice Problems 6: Answers

1. Let $y = x^4 - x^3 - x + 1$. Find the value of x where y has its absolute minimum.

Answer. Since y is a polynomial in x of even degree, $y \to \infty$ as $x \to \pm\infty$, so there is a point at which y has an absolute minimum. To find it, we differentiate:

$$y' = 4x^3 - 3x^2 - 1$$

and solve for $y' = 0$. $x = 1$ is one root; by long division we find $4x^3 - 3x^2 - 1 = (x-1)(4x^2 + x + 1)$, and the quadratic factor has no real roots. Thus $x = 1$ is the only place where the function has a horizontal tangent, so is the value at which y has an absolute minimum.

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2. Find all points of local maxima and minima of the function $f(x) = x(4 + x^{-2})$.

Answer. Differentiate:

$$f'(x) = 4 + x^{-2} + x(\frac{-2}{x^3}) = 4 - x^{-2} \ .$$

We have $f'(x) = 0$, when $x^2 = 1/4$, or $x = \pm 1/2$. Since $f'(x)$ is positive for $|x|$ very large, and is negative for $|x|$ very small, f is decreasing as we approach 1/2 from the left, and increasing as we leave 1/2 to the right. Thus f has a local minimum at $x = 1/2$. Since f is an odd function, it has a local maximum at $x = -1/2$. Notice that f does not have a global maximum or minimum, since

$$\lim_{x \to 0^-} f(x) = -\infty \ , \quad \lim_{x \to 0^+} f(x) = \infty \ .$$

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3. Find the absolute maxima and minima of the function

$$f(w) = w\sqrt{w+1}$$

on the interval $-1 \leq w \leq 4$.

Answer. Differentiate:

$$f'(w) = (w+1)^{1/2} + \frac{1}{2}w(w+1)^{-1/2} = \frac{\frac{3}{2}w + 1}{(w+1)^{1/2}} \ ,$$

which is 0 when $w = -2/3$. The points to check are this and the endpoints: -1,4. $f(-1) = 0$, $f(-2/3) = -(2/3)\sqrt{1/3}$, $f(4) = 4\sqrt{5}$, so the minimum is $-2/(3\sqrt{3})$ at $x = -2/3$, and the maximum is $4\sqrt{5}$ at $x = 4$.

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4. Find the maximum and the minimum of $y = x\sqrt{1-x^2}$ on the interval $-1 \leq x \leq 1$.
Answer. Differentiate:

$$\frac{dy}{dx} = \sqrt{1-x^2} + x\frac{-2x}{2\sqrt{1-x^2}} = \frac{1-2x^2}{(1-x^2)^{3/2}}$$

This is zero when $x = \pm 1/\sqrt{2}$. The critical values are $-1, -1/\sqrt{2}, 1/\sqrt{2}, 1$, and the corresponding values of y are 0, $-[(1/2)\sqrt{3/2}]$, $[(1/2)\sqrt{3/2}]$, 0, so the nonzero values are the minimum and maximum repsectively.

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5. Let $y = \sin^2 x + \cos x$, for x in the interval $[-\pi, \pi]$. Find the absolute maximum and minimum of y.

Answer. Differentiate:

$$y' = 2\sin x \cos x - \sin x = \sin x(2\cos x - 1)\ .$$

This is zero at $x = -\pi, 0, pi$ and $x = \pm\pi/3$. The values of y at these points are, respectively $-1, 5/4, 1, 5/4, -1$ Thus the absolute maximum is 5/4, and the absolute minimum is -1. Note that at $x = 0$ we have a local minimum.

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6. Let $y = \dfrac{x}{x^2 - 4x + 3}$. Find the intervals in which y is increasing; in which y is decreasing.

Answer. Differentiate the function:

$$y' = \frac{x^2 - 4x + 3 - x(2x - 4)}{(x^2 - 4x + 3)^2} = \frac{-x^2 + 3}{(x^2 - 4x + 3)^2}\ .$$

Since the denominator is a square, the sign is determined by the numerator. When $x^2 < 3$, y' is positive, and when $x^2 > 3$ it is negative. Thus y is decreasing in the intervals $(-\infty, -\sqrt{3})$, $(\sqrt{3}, \infty)$, and is increasing in the interval $(-\sqrt{3}, \sqrt{3})$.

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7. For what number x between 0 and 1 is $x^{1/3} - x$ a maximum?

Answer. Let $y = x^{1/3} - x$. We see that $y = 0$ at $x = 0$ and 1, and at $x = 1/8$, $y = 1/2 - 1/8 > 0$, so there is a maximum between 0 and 1. To find it, we differentiate:

$$y' = \frac{1}{3}x^{-2/3} - 1\ .$$

This is zero at $x = 3^{-3/2}$, so this is the only critical point, and thus must be the maximum point.

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8. Show that the equation $2x^{12} - 3x^6 + x = 0$ has a root strictly between 0 and 1.

Answer. Let $f(x) = 2x^{12} - 3x^6 + x = 0$. we have $f(0) = 0$ and $f(1) = 0$. Differentiating, we find $f'(x) = 24x^{11} - 18x^5 + 1$, so $f'(0) > 0$, and $f'(1) > 0$. Thus f is increasing at both endpoints, so that f has positive values just to the right of 0, and negative values just to the left of 0. By the intermediate value theorem then, there is some point c strictly between 0 and 1 at which $f(c) = 0$.