Calculus I
Practice Problems 5: Answers

1. A curve is given by the equation $x^2 - xy + y^2 = 7$. Find the equation of the line tangent to this curve at the point (2,-1).

Answer. Take the differential of the equation:

$$2xdx - xdy - ydx + 2ydy = 0 \; .$$

At $x = 2,\; y = -1$ this becomes $4dx - 2dy + dx - 2dy = 0$, or $5dx - 4dy = 0$. Along the tangent line we replace dx and dy by $x - 2$ and $y - (-1)$, giving the equation $5(x-2) - 4(y+1) = 0$, or $5x - 4y = 14$.

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2. Find the slope of the curve defined by the relation $4(x^2 + xy) = 2y^3 - y^2$ at the point $(1,2)$.

Answer. Differentiate:

$$4\left(2x + y + x\frac{dy}{dx}\right) = \left(6y^2 - 2y\right)\frac{dy}{dx}$$

At $(1,2)$:

$$4\left(2 + 2 + \frac{dy}{dx}\right) = \left(6 \cdot 2^2 - 2 \cdot 2\right)\frac{dy}{dx}$$

$$16 + 4\frac{dy}{dx} = 20\frac{dy}{dx}$$

so $\dfrac{dy}{dx} = 1$.

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3. Variables x and y are related by the formula $x\sin y + y\sin x = \pi$. If $dy/dt = 3$ when $x = 3\pi/2$ and $y = \pi/2$, what is dx/dt?

Answer. Differentiate with respect to t:

$$\frac{dx}{dt}\sin y + x\cos y\frac{dy}{dt} + \frac{dy}{dt}\sin x + y\cos x\frac{dx}{dt} = 0 \; .$$

Put in the given values:

$$\frac{dx}{dt}(1) + \frac{3\pi}{2}(0)(3) + 3(-1) + \frac{\pi}{2}(0)x\frac{dx}{dt} = 0 \; ,$$

or $dx/dt = 3$.

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4. The relation $\cos y + x = \sin y$ determines a curve in the x-y plane. Find the slope of the line tangent to the curve at the point $(1, \pi/2)$

Answer. Differentiate: $-(\sin y)y' + 1 = (\cos y)y'$. Substitute $x = 1,\; y = \pi/2:\;\; -y' + 1 = 0$. Thus the slope is $y' = 1$.

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5. Consider the curve given by the equation: $y^2+xy+x^2=1$. At what points does this curve have a horizontal tangent line?

Answer. Differentiate with respect to x:

$$2y\frac{dy}{dx}+x\frac{dy}{dx}+y+2x=0\ ,$$

so $dy/dx=0$ when $y+2x=0$, or $y=-2x$. Substitute this in the given equation: $(-2x)^2+x(-2x)+x^2=1$, or $3x^2=1$. Thus $x=\pm 1/\sqrt{3}$, and the desired points are $(1/\sqrt{3},-2/\sqrt{3})$, $(-1/\sqrt{3},2/\sqrt{3})$.

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6. Consider the curve given by the equation: $x^2y-y^3=1$. At what points does this curve have a vertical tangent line?

Answer. Taking the differential of the defining equation we have

$$2xydx+x^2dy-3y^2dy=0\ .$$

Now, the tangent line is vertical at points where the infinitesimal change in x is 0; that is $dx=0$. Thus we have to solve $x^2dy-3y^2dy=0$, or $x^2=3y^2$. Substituting this in the defining equation we get: $3y^3-y^3=1$, so $y=2^{-1/3}$, and thus $x^2=3\left(2^{-2/3}\right)$, and the answer is: the points

$$\left(\frac{3^{1/2}}{2^{1/3}},\frac{1}{2^{1/3}}\right)\ ,\quad \left(-\frac{3^{1/2}}{2^{1/3}},\frac{1}{2^{1/3}}\right)\ .$$

Note that the tangent line is horizontal when $dy=0$, so we could have solved problem 5 using differentials as well.

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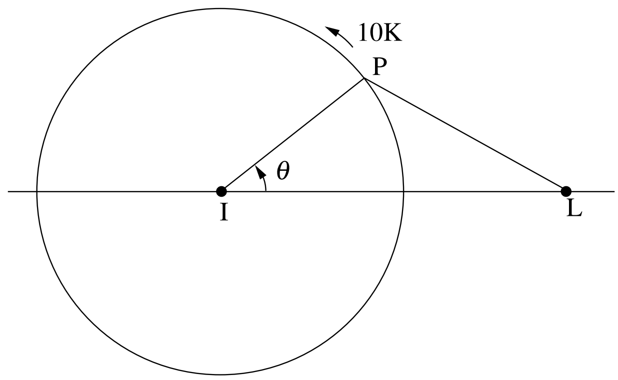
7. A ship is traveling in a circle of radius 6 nautical miles around an island at a speed of 10 knots. A lighthouse is 10 miles due east of the island. At what rate is the distance between ship and lighthouse increasing when the ship is exactly due north of the island?

Answer. Draw the diagram below, where P is a typical position of the ship, I is the island, and L is the lighthouse.



Let s represent distance traveled by the ship; this is the length of arc along the circle. We are given that $ds/dt=10$ nautical miles per hour. Let θ be the angle at I between the ship's position and the due east direction. Then since "arc length = (radius)(angle)", we have $\theta=s/6$, so $d\theta/dt=(ds/dt)/6=10/6$ radians per hour. Now, using coordinates, the position of the ship is $(6\cos\theta, 6\sin\theta)$, and the lighthouse is at $(10,0)$. Thus, for x the distance between ship and lighthouse, we have

$$x^2 = (6\cos\theta - 10)^2 + (6\sin\theta)^2 \ = 100 - 120\cos\theta \ .$$

Differentiating,

$$2x\frac{dx}{dt} = -120(-\sin\theta)\frac{d\theta}{dt} = 120(5/3)\sin\theta \ .$$

Now, when the ship is directly north of the island, $\theta = \pi/2$ and $x = \sqrt{10^2+6^2} = \sqrt{136}$. Thus

$$\frac{dx}{dt} = \frac{5/3(120)}{2\sqrt{136}} = 8.575 \text{ knots} \ .$$

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8. A new stadium, built like a cylinder capped with a hemispherical dome is proposed to have a diameter of 500 feet. To include another 2000 seats, the diameter must be increased by 10 feet. By approximately how much will the area of the dome be increased? (Note: the area of a sphere of radius r is $4\pi r^2$.)

Answer. The area of the dome is half the area of the sphere, so $A = 2\pi r^2$. Taking differentials, $dA = 4\pi r dr$. When $r = 250$, if the increment in r is $dr = 10$, we then have $dA = 4\pi(250)(10) = 31,415$ sq. ft.

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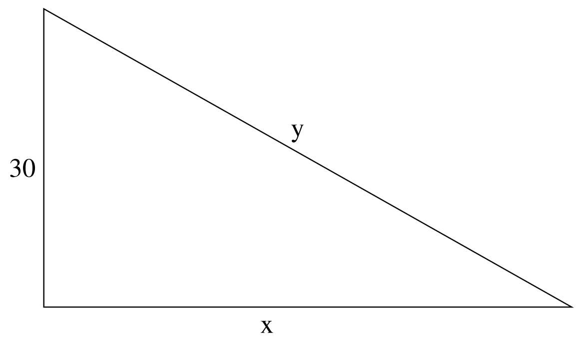
9. A cat is walking toward a telephone pole of height 30 feet. She is walking at a steady rate of 4 ft/sec. A bird is perched on top of the telephone pole. When the cat is 45 feet from the base of the pole, at what rate is the distance between bird and cat decreasing?

Answer. Draw the figure:



The variables are the distance x of the cat from the base of the pole, and the distance y of the cat from the bird. The variables are related by the Pythagorean theorem: $y^2 = x^2 + (30)^2$. Differentiate to relate the rates of change:

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt} \ .$$

Now at $x = 45$, we find $y^2 = 45^2 + 30^2$, so $y = 15\sqrt{13}$. Putting this value, and $dx/dt = 4$ ft/sec in the preceding equation, we find

$$\frac{dy}{dt} = \frac{12}{\sqrt{13}} \quad \text{ft/sec.}$$

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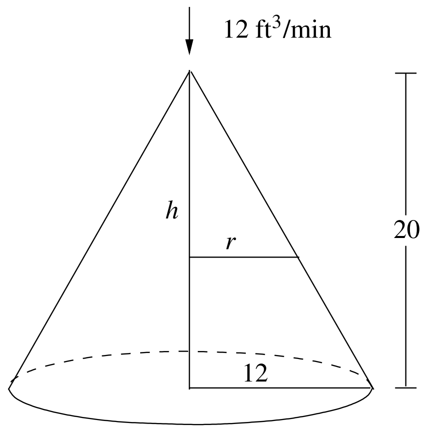
10. Water is flowing into a conical tank through an opening at the vertex at the top at the rate of 12 cu. ft./min. The base of the tank is a circle of radius 12 ft. and the height of the cone is 20 ft. At what rate is the water level rising when the water level is 4 ft. from the top? The formula for the volume of a cone of base radius r and height h is $V = (1/3)\pi r^2 h$.

Answer. Draw the figure:



The variables are time t, volume of the part of the cone without water V, height of the water surface from the vertex of the cone h, and radius r of the circle of the water surface. We have the equations

$$V = \frac{1}{3}\pi r^2 h\ , \quad \frac{r}{12} = \frac{h}{20}\ .$$

The second equation comes from the comparison of similar triangles. Use it to express V in terms of h alone:

$$V = \frac{1}{3}\pi\left(\frac{3}{5}h\right)^2 h = \frac{3\pi}{25}h^3\ .$$

Differentiate with respect to time:

$$\frac{dV}{dt} = \frac{9\pi}{25}h^2\frac{dh}{dt}\ .$$

Now put in dV/dt = -12, h = 4 (remember: V is the volume of the empty part of the cone, and h is its depth):

$$-12 = \frac{9\pi}{25}16\frac{dh}{dt}\ ,$$

so $dh/dt = -25/12\pi$ ft/min. Thus the water is rising at the rate of $25/12\pi$ ft/min.