**Calculus I**
**Practice Problems 12: Answers**

1. Kansas can be modelled as a rectangle of length 500 miles and of height 300 miles. The population density decreases as one moves west through the state; in fact the density $x$ miles west of the eastern border is about

$$\delta(x) = 40 - 35\left(\frac{x}{500}\right)^2$$

people per square mile. About what is the population of Kansas?

**Answer**. The population in a strip of width $dx$ at a distance $x$ from the eastern border is $dP = \delta(x)A(x)$ where $A(x) = 300dx$ is the area of the strip. We find the total population by integrating:

$$P = \int_0^{500}\left(40 - 35\left(\frac{x}{500}\right)^2\right)300dx = 300\left(40x - \frac{35}{500^2}\frac{x^3}{3}\right)\Bigg|_0^{500} = 4.25 \text{ million} .$$

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2. A pencil sharpener is made by drilling a cone out of a sphere; the cone has as its axis a diameter of the sphere, and its vertex is on the surface of the sphere. If the ratio of the height to base radius in the cone is 4 to 1, and the sphere has a 1 inch radius, what is the volume of the pencil sharpener?

**Answer**. We can model this as a solid of revolution as follows. The sphere is obtained by rotating the curve $y = \sqrt{1-x^2}$ around the $x$-axis. Let's put the vertex of the cone at (-1,0) and its axis along the $x$-axis. The given information tells us that the cone is generated by rotating (around the $x$-axis) the line of slope 1/4 and $x$-intercept -1. The equation of this line is $y = (x+1)/4$ (see the figure). Now, we calculate the volume using the method of washers:

$$dV = \left(\pi R^2 - \pi r^2\right)dx = \pi\left[(1-x^2) - \frac{(1+x)^2}{16}\right]dx$$

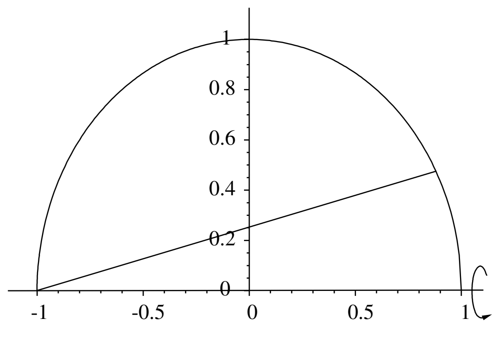
The range of integration is from -1 to the $x$-coordinate of the point of intersection of the two curves. To find that, we solve

$$\sqrt{1-x^2} = \frac{(1+x)}{4} \quad \text{which simplifies to} \quad 17x^2 + 2x - 15 = 0 ,$$

which has the solutions $x = -1,\ 15/17$. Thus the volume is

$$\int_{-1}^{15/17}\pi\left[(1-x^2) - \frac{(1+x)^2}{16}\right]dx = \frac{1024}{867}\pi = 3.71$$

cubic inches. Should a problem like this occur on an examination it suffices to end the argument with the definite integral to be computed. In cases like this, the actual computation is arithmetically tedious, and is best done on the computer (I used MAPLE).



3. An ellipsoid is formed by rotating the curve $4x^2 + y^2 = 1$ around the $x$-axis. What is its volume? What is the volume of the ellipsoid obtained by rotating this curve about the $y$-axis?

**Answer**. By symmetry, the volume is twice the volume of the solid obtained by rotating the region in the first quadrant bounded by the curve $y = \sqrt{1-4x^2}$.

a) For rotation about the $x$-axis we use the disc method. We have $dV = \pi y^2 dx = \pi(1-4x^2)dx$. To find the range of $x$ we solve the equation $1-4x^2 = 0$; this gives $x = 1/2$. Thus

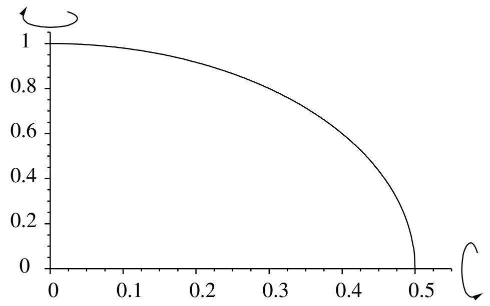
$$Volume = 2\int_0^{1/2} \pi\left(1-4x^2\right)dx = 2\pi\left[x - \frac{4}{3}x^3\right]\Big|_0^{1/2} = 2\pi\left[\frac{1}{2} - \frac{4}{3}\left(\frac{1}{8}\right)\right] = \frac{2\pi}{3}\,.$$

b). To rotate about the $y$-axis we use the shell method: $dV = 2\pi xydx = 2\pi x\sqrt{1-4x^2}dx$. Thus

$$Volume = 2\int_0^{1/2} 2\pi x\sqrt{1-4x^2}dx\,.$$

Let $u = 1-4x^2$, $du = -8xdx$. Then when $x = 0$, $u = 1$ and when $x = 1/2$, $u = 0$. Thus

$$Volume = -\frac{\pi}{2}\int_1^0 u^{1/2}du = -\frac{\pi}{2}\frac{2}{3}u^{3/2}|_1^0 = \frac{\pi}{3}\,.$$



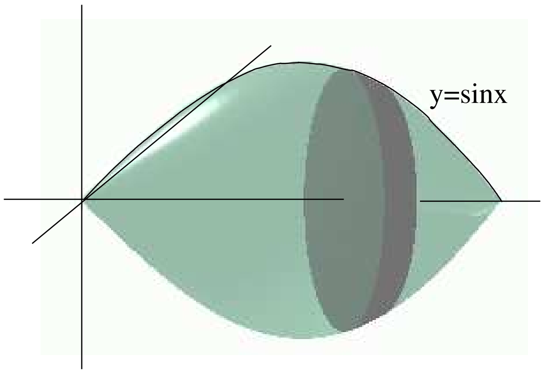

4. Consider the region in the first quadrant bounded by $y = \sin x$, $x = 0$, $x = \pi$. Find the volume of the solid obtained by rotating this region about the $x$-axis. Hint: $\cos(2x) = 1 - 2\sin^2 x$.

**Answer**. We use the disc method: $dV = \pi y^2 dx = \pi \sin^2 xdx$. Thus the volume is

$$\int_0^\pi \sin^2 xdx = \frac{1}{2}\int_0^\pi (1-\cos(2x))dx\,,$$

by the hint. This gives us

$$Volume \;=\; \frac{1}{2}[x - \frac{\sin(2x)}{2}]_0^\pi = \frac{\pi}{2}\,.$$




5. Consider the region in the first quadrant bounded by $y = \sin x$, $x = 0$, $x = \pi$. Find the volume of the solid obtained by rotating this region about the $x$-axis. Hint: The derivative of $\sin x - x\cos x$ is $x\sin x$.

**Answer**. We use the shell method: $dV = 2\pi xy dx$. The volume is

$$\int_0^\pi x\sin x dx = \big[\sin x - x\cos x\big]_0^\pi = \pi \ ,$$

by the hint.

6. Find the length of the curve $y = t^3$, $x = t^2$, $0 \le t \le 1$.

**Answer**. $dy = 3t^2 dt$, $dx = 2t dt$, so $ds^2 = (4t^2 + 9t^4)dt^2$, and thus

$$ds = t\sqrt{4+9t^2}$$

$$L = \int_0^1 t\sqrt{4+9t^2}dt = \frac{1}{18}\int_4^{13} u^{1/2} du$$

which comes out to $(1/27)[13^{3/2} - 8]$. We made the substitution $u = 4 + 9t^2$, $du = 18t dt$.

7. The equations $x = e^{at}\cos t$, $y = e^{at}\sin t$ define the logarithmic spiral. Find the length of this curve for $0 \le t \le 2\pi$.

**Answer**. Differentiating

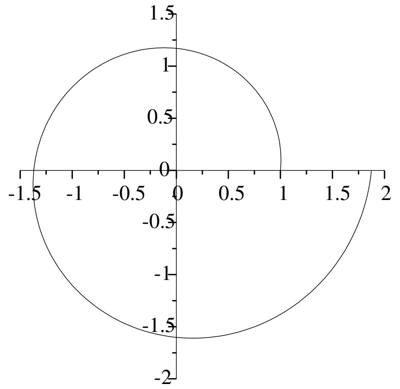
$$dx = (ae^{at}\cos t - e^{at}\sin t)dt \ , \quad dy = (ae^{at}\sin t + e^{at}\cos t)dt \ .$$

This gives

$$ds^2 = dx^2 + dy^2 = e^{2at}((a\cos t - \sin t)^2 + (a\sin t + \cos t)^2)dt^2 = e^{2at}(a^2+1)dt^2 \ .$$

Thus the length is

$$\int_0^{2\pi} ds = \int_0^{2\pi} e^{at}\sqrt{a^2+1}dt = \sqrt{a^2+1}\big(\frac{e^{at}}{a}\big)_0^{2\pi} = \frac{\sqrt{a^2+1}}{a}(e^{2\pi a} - 1) \ .$$




8. Find the integral giving the length of the curve $y = \sqrt{1+x^2}$ for $0 \leq x \leq 1$.

**Answer**. Differentiate:

$$\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}} ,$$

so

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x}{\sqrt{1+x^2}}\right)^2 = \frac{1+2x^2}{1+x^2} .$$

The integral giving the arc length is thus

$$\int_0^1 \sqrt{\frac{1+2x^2}{1+x^2}} dx .$$

9. Find the area of the surface obtained by rotating the line segment $y = 3x-3$, $3 \leq x \leq 5$ about the $x$-axis.
**Answer**. Here $ds^2 = dx^2 + dy^2 = dx^2 + (3dx)^2 = 10dx^2$, so $ds = \sqrt{10}dx$. Then the length of the curve is

$$Length = \int_3^5 \sqrt{10}dx = 2\sqrt{10} .$$

The centroid of the line is its midpoint, so is at $(4,9)$. Thus, by Pappus' theorem, the surface area is $2\pi(4)(2\sqrt{10} = 16\pi\sqrt{10}$. If, instead of using Pappus' theorem, we compute directly, we have $dS = 2\pi x ds = 2\pi\sqrt{10}xdx$, so

$$Surface\ Area = 2\pi\sqrt{10}\int_3^5 xdx = 2\pi\sqrt{10}(8) .$$

10. Find the area of the bowl obtained by rotating the parabola $y = x^2$, $0 \leq x \leq a$ about the $x$-axis.

**Answer**. Here we obtain $ds^2 = dx^2 + dy^2 = dx^2 + (2xdx)^2 = \left(1+4x^2\right)dx^2$, so $ds = \sqrt{1+4x^2}dx$ and $dS = 2\pi x\sqrt{1+4x^2}dx$. Thus

$$Surface\ Area = 2\pi\int_0^a x\sqrt{1+4x^2}dx = 2\pi\left(\frac{1}{8}\right)\left(1+4x^2\right)^{3/2}\Big|_0^a = \frac{\pi}{4}\left(\sqrt{1+4a^2}-1\right) .$$