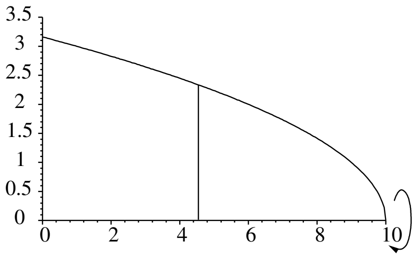
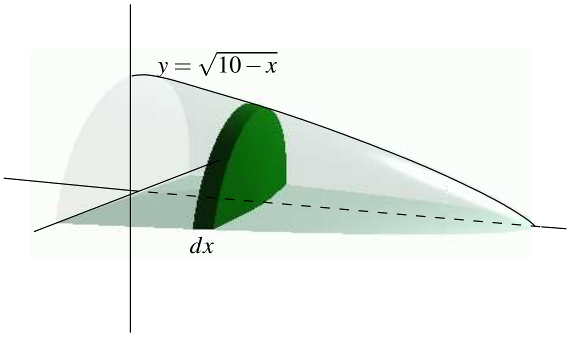
Calculus I
Problem Set 11 Answers

1. A solid is formed over the region in the first quadrant bounded by the curve $y = \sqrt{10-x}$ so that the section by any plane perpendicular to the x-axis is a semicircle. What is the volume of this solid?

Answer. We sweep out along the x-=axis. The section at x is a semicircle of radius $y/2$, so has area $A(x) = (\pi/2)(y/2)^2 = (\pi/8)(10-x)$. Thus

$$V = \frac{\pi}{8}\int_0^{10}(10-x)dx = \frac{\pi}{8}\big(10x - \frac{x^2}{2}\big)\Big|_0^{10} = \frac{25\pi}{4}\ .$$

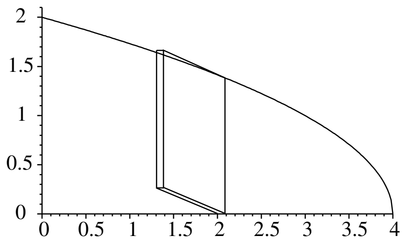


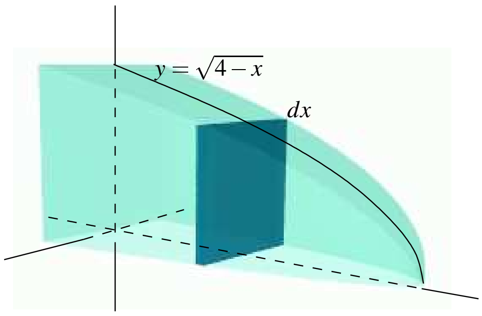


2. A solid is formed over the region in the first quadrant bounded by the curve $y = \sqrt{4-x}$ so that the section by any plane perpendicular to the x-axis is a square. What is the volume of this solid?

Answer. The section at x has area $y^2 = 4-x$, so

$$V = \int_0^4 (4-x)dx = 8\ .$$





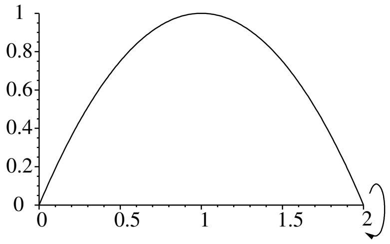
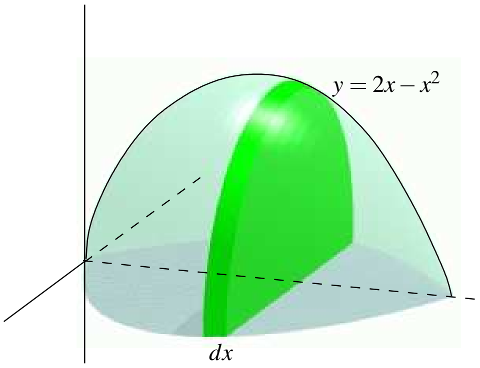
3. A solid is formed over the region in the first quadrant bounded by the curve $y = 2x - x^2$ so that the section by any plane perpendicular to the x-axis is a semicircle. What is the volume of this solid?

Answer. As in problem 1,

$$dV = \frac{\pi}{2}(\frac{y}{2})^2 = \frac{\pi}{8}(2x - x^2)^2 dx = \frac{\pi}{8}(4x^2 - 4x^3 + x^4)dx \; .$$

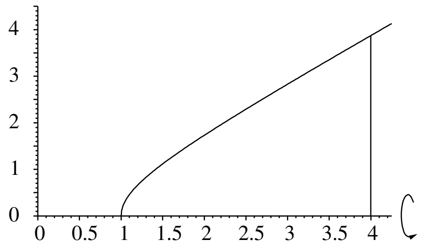
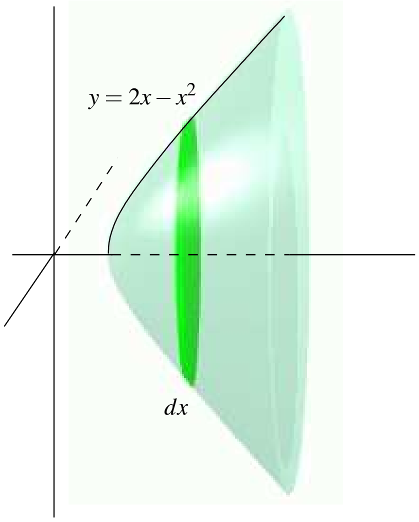
Integrating dV from 0 to 2, we get

$$V = \frac{\pi}{8}(\frac{32}{3} - 16 + \frac{32}{5}) = \frac{2\pi}{15} \; .$$

4. The region in the first quadrant bounded by $y = \sqrt{x^2 - 1}$, $y = 0$, $x = 1$, $x = 4$ is revolved around the x-axis. Find the volume of the resulting solid.

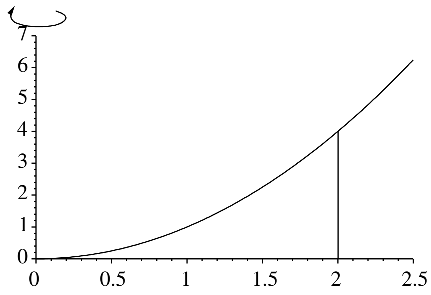
Answer. Here we find that at a typical x between 1 and 4, $dV = \pi r^2 dx = \pi(x^2 - 1)dx$. Integrating, we get $V = 18\pi$.

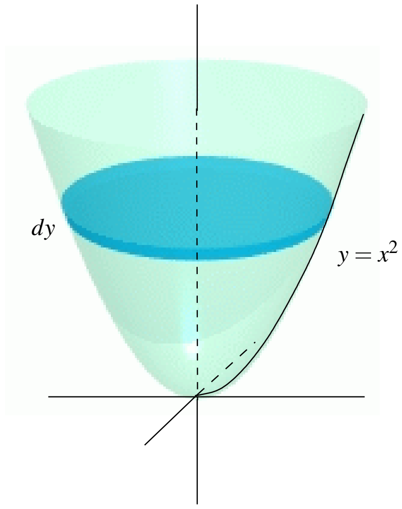



5. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = x^2$, $x = 2$ and the x-axis.

Answer. Here we will use the washer method, sweeping out along the y-axis, with y ranging from 0 to 4. At a typical y, $dV = (\pi R^2 - \pi r^2)dy$, and $R = 2,\ r = \sqrt{y}$. Thus the volume is

$$V = \pi \int_0^4 (4-y)dy = 8\pi\ .$$





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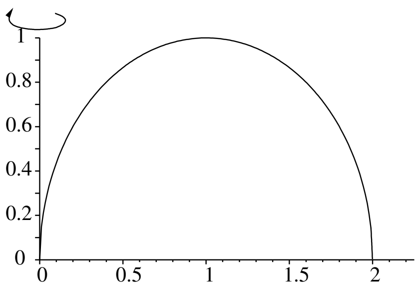
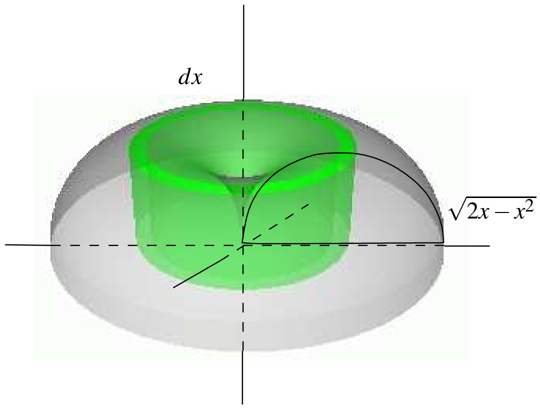
6. The region in the first quadrant under the curve $y = 2x - x^2$ is rotated about the y-axis. Find the volume of the resulting solid.

Answer. Here we sweep out along the x-axis from $x = 0$ to $x = 2$, using the shell method. At a typical x, $dV = 2\pi xy dx = 2\pi(2x^2 - x^3)dx$, and

$$V = 2\pi \int_0^2 (2x^2 - x^3)dx = 2\pi(\frac{2x^3}{3} - \frac{x^4}{4})|_0^2 = \frac{8\pi}{3}\ .$$





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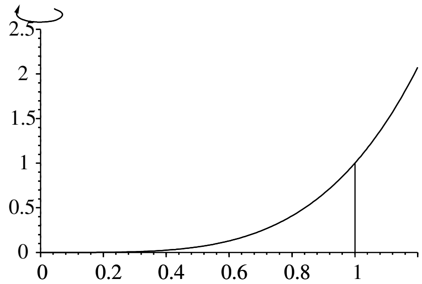
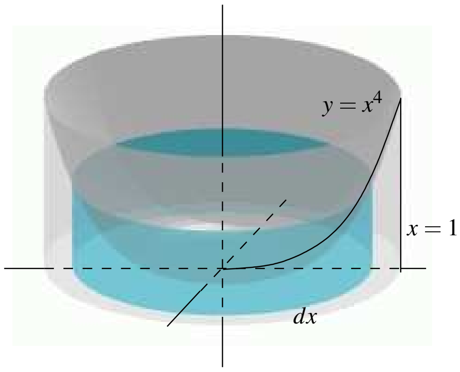
7. The region in the first quadrant bounded by $y = x^4$ and $x = 1$ is revolved around the y-axis. Find the volume of the resulting solid.

Answer. Here let's use the shell method, sweeping out along the x-axis (compare this with problem 5). $dV = 2\pi xy dx = 2\pi x^5 dx$. Integrating from 0 to 1, we get $V = \pi/53$.





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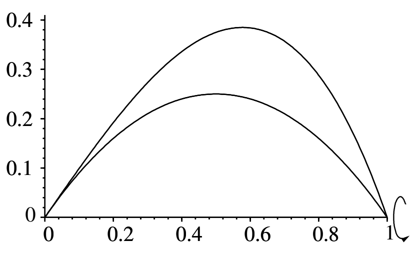
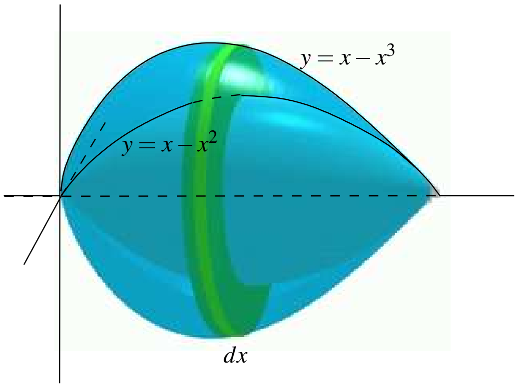
8. The region in the first quadrant bounded by $y = x - x^2$ and $y = x - x^3$ is revolved around the x-axis. Find the volume of the resulting solid.

Answer. Using the washer method, at a typical x between 0 and 1, $dV = (\pi R^2 - \pi r^2)dx = \pi[(x-x^3)^2 - (x-x^2)^2]dx$. After some algebra, we obtain

$$V = \int_0^1 (2x^3 - 3x^4 + x^6)dx = \pi(\frac{1}{2} - \frac{3}{5} + \frac{1}{7}) = .0428\pi \ .$$





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9. The *average value* of a function $y = f(x)$ defined over an interval $[a,b]$ is defined to be

$$y_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)dx \ .$$

Find the average of $y = \sin x$ over the interval $[0, \pi]$.

$$y_{\text{ave}} = \frac{1}{\pi}\int_0^{\pi} \sin x dx \; = \frac{1}{\pi}(-\cos x)|_0^{\pi} = \frac{2}{\pi}.$$

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10. Let $g(x) = x^2 + x^3$ for x in the interval $0 \le x \le 10$. Find the average, or mean, value of g on the interval. Find the average slope of the graph of $y = g(x)$ on the interval.

Answer. The average value of the function is

$$\frac{1}{10}\int_0^{10}(x^2 + x^3)dx = \frac{1}{10}[\frac{x^3}{3} + \frac{x^4}{4}]_0^{10} = \frac{100}{3} + \frac{1000}{4} = 283.33\ .$$

Since the slope is $y' = g'(x)$m the average slope is

$$\frac{1}{10}\int_0^{10} g'(x)dx = \frac{1}{10}(g(10) - g(0)) = 110\ .$$