Calculus I
Practice Problems 10: Answers

1. $\int_1^3 (2t+1)^3\,dt =$

Answer.

$$\int_1^3 (2t+1)^3 dt = \frac{1}{2}\int_3^7 u^3 du = \frac{1}{2}\frac{u^4}{4}\Big|_3^7 = \frac{1}{8}(7^4 - 3^4)$$

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2. $\int_{-1}^1 (4x^3 - 2x^2 + 1)dx =$

Answer. Since x^3 is an odd function and the domain is symmetric about 0, the first term contributes nothing. Thus the integral is equal to

$$\int_{-1}^1 (-2x^2+1)dx = (-2x^3/3 + x)\Big|_{-1}^1 = \frac{2}{3}$$

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3. Calculate the definite integrals:

a) $\int_{-4}^4 (x^2 - 3 + \cos x)dx$

Answer. Since this is an even function and the domain is symmetric about 0, the integral is

$$2\int_0^4 (x^2 - 3 + \cos x)dx = 2\Big[\frac{x^3}{3} - 3x + \sin x\Big]_0^4 = 2\Big[\frac{64}{3} - 12 + \sin(4)\Big] = \frac{56}{3} + 2\sin(4)\ .$$

b) $\int_0^{\pi/4} \frac{\sin x}{\cos^3 x} dx$

Answer. Let $u = \cos x,\ du = -\sin x dx$. When $x = 0, u = 1$ and when $x = \pi/4,\ u = \sqrt{2}/2$. Thus

$$\int_0^{\pi/4} \frac{\sin x dx}{\cos^3 x} = -\frac{1}{2}\int_1^{\sqrt{2}/2} u^{-3} du = \frac{1}{2}u^{-2}\Big|_1^{\sqrt{2}/2} = \frac{1}{2}\Big(\frac{1}{2/4} - 1\Big) = \frac{1}{2}\ .$$

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4. Integrate:

a) $\int_1^4 \frac{1}{\sqrt{y}(\sqrt{y}+1)^2} dy$

Answer. Let $u = y^{1/2},\ \ du = (1/2)y^{-1/2}dy$. When $y = 1,\ u = 1$ and when $y = 4,\ u = 2$. Thus

$$\int_1^4 \frac{1}{\sqrt{y}(\sqrt{y}+1)^2} dy = 2\int_1^2 \frac{du}{(u+1)^2} = -2(u+1)^{-1}\Big|_1^2 = -2\Big[\frac{1}{3} - \frac{1}{2}\Big] = \frac{1}{3}$$

b) $\int_0^{\pi/2} \cos^2 x \sin x dx =$

Answer. $= -\frac{\cos^3 x}{3}\Big|_0^{\pi/2} = \frac{1}{3}$

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5. Evaluate

a) $\dfrac{d}{dx}\displaystyle\int_0^{2x+1} \cos t\,dt$

Answer. Let $u = 3x+1$. By the fundamental theorem of the calculus $d/du \int_0^u \cos t\,dt = \cos u$. Now, by the chain rule

$$\frac{d}{dx}\int_0^{2x+1} \cos t\,dt = \left(\frac{d}{du}\int_0^u \cos t\,dt\right)\left(\frac{du}{dx}\right) = (\cos u)(2) = 2\cos(2x+1)\ .$$

b) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} t^3 dt$

Answer. Let $u = x^2$. By the fundamental theorem of the calculus $d/du \int_0^u t^3 dt = u^3$. Now, by the chain rule

$$\frac{d}{dx}\int_0^{x^2} t^3 dt = \left(\frac{d}{du}\int_0^u t^3 dt\right)\left(\frac{du}{dx}\right) = u^3(2x) = (x^2)^3(2x) = 2x^7\ .$$

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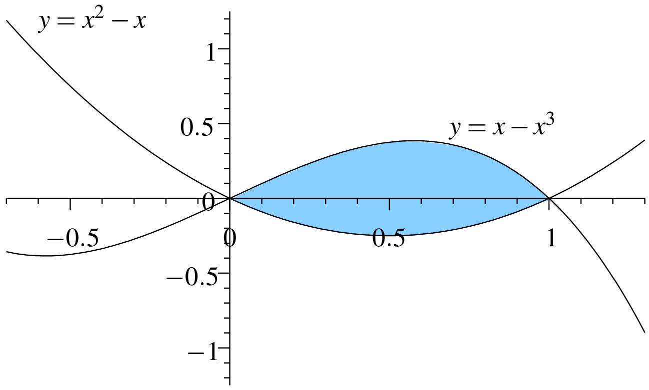
6. Find the area of the region in the right half plane ($x > 0$) bounded by the curves $y = x - x^3$ and $y = x^2 - x$.

Answer. First, we find the points of intersection of the curves by solving the equation $x - x^3 = x^2 - x$. This becomes $x^3 + x^2 - 2x = 0$, which has the solutions $x = -2, 0, 1$. Since we are interested only in $x > 0$, the range of integration is the interval $(0,1)$. From the graph (see the figure), the cubic curve lies above the quadratic, so the area is

$$\int_0^1 [(x-x^3)-(x^2-x)]dx = \int_0^1 (-x^3 - x^2 + 2x)dx = -\frac{1}{4} - \frac{1}{3} + 1 = \frac{5}{12}\ .$$



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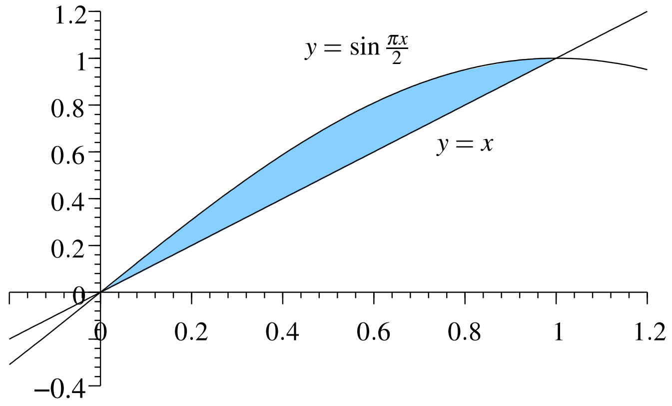
7. Find the area of the region in the first quadrant bounded by the curves $y = \sin \frac{\pi}{2}x$ and $y = x$.

Answer. The curves intersect at $x = 0,\ 1$, and the sine curve is above the line (see the figure), so the area is

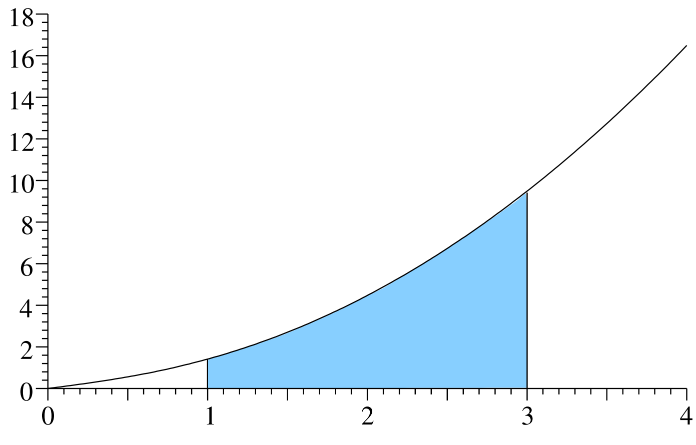
$$\int_0^1 (\sin\frac{\pi}{2}x - x)dx = \frac{2}{\pi}(-\cos\frac{\pi}{2}x) - \frac{x^2}{2}\Big|_0^1 = (\frac{2}{\pi}(0) - \frac{1}{2}) - (\frac{2}{\pi}(-1) - 0) = \frac{2}{\pi} - \frac{1}{2}$$



8. Find the area of the region under the curve $y = x\sqrt{x^2+1}$, above the x-axis and bounded by the lines $x = 1$ and $x = 3$.

Answer. The area (see the figure) is given by $\int_1^3 x\sqrt{x^2+1}dx$. Let $u = x^2+1,\ du = 2xdx$. When $x = 1,\ u = 2$ and when $x = 3,\ u = 10$. This substitution leads to:

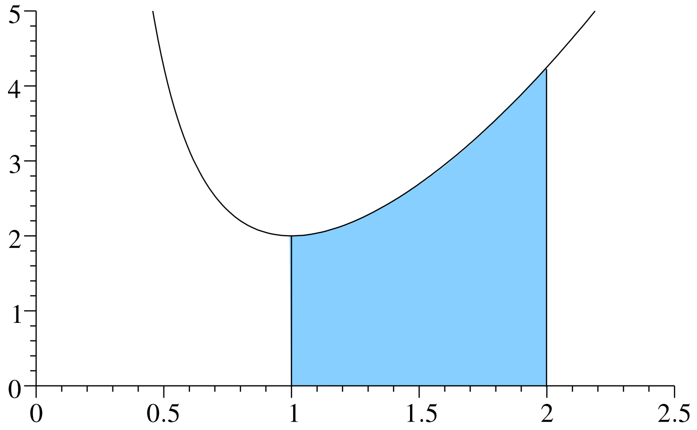
$$\int_1^3 x\sqrt{x^2+1}dx = \frac{1}{2}\int_2^{10} u^{1/2}du = \frac{1}{2}\frac{2}{3}u^{3/2}|_2^{10} = \frac{1}{3}(10\sqrt{10} - 2\sqrt{2}) .$$



9. Find the area under the curve $y = x^2 + x^{-2}$, above the x-axis and between the lines $x = 1$ and $x = 2$.

Answer. The area is

$$\int_1^2 (x^2 + x^{-2})dx = (\frac{x^3}{3} - x^{-1})|_1^2 = (\frac{8}{3} - \frac{1}{2}) - (\frac{1}{3} - 1) = \frac{17}{6}$$



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10. What is the area of the region bounded by the curves $y = x^3 - x$ and $y = 3x$?

Answer. First find the points of intersection:

$$x^3 - x = 3x \quad \text{or} \quad x^3 = 4x$$

has the solutions $x = -2, 0, 2$. The line $y = 3x$ lies below the curve $y = x^3 - x$ in the interval (-2,0) and above that curve in the interval (0,2) (see the accompanying figure). The areas of these two regions are given by the integrals:

$$\int_{-2}^{0} [(x^3 - x - 3x)]dx\,, \quad \int_{0}^{2} [3x - (x^3 - x)]dx\,.$$

Since the two intervals are symmetric about 0, and the integrand is an odd function, these two integrals are the same. Thus the area is

$$2\int_{0}^{2} [3x - (x^3 - x)]dx = 2\int_{0}^{2} (4x - x^3)dx = 2(2x^2 - \frac{x^4}{4})\Big|_0^2 = 2(8 - 16/4) = 8\,.$$