Calculus I
Practice Final Exam, Answers

1. Find the derivatives of the following functions:

a) $f(x) = (x^3-1)(x^2+1)^2$

Answer. $f'(x) = (x^3-1)2(x^2+1)2x + 3x^2(x^2+1)^2$

$$= (x^2+1)[4x(x^3-1) + 3x^2(x^2+1)] = (x^2+1)[7x^4+3x^2-4x]$$

b) $g(x) = \dfrac{\sin x}{\cos x + 1}$

Answer. $g'(x) = \dfrac{(\cos x+1)\cos x - \sin x(-\sin x)}{(\cos x+1)^2} = \dfrac{1}{\cos x+1}$

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2. Find the derivatives of the following functions:

a) $f(x) = \sin^3(4x+1)$

Answer. $f'(x) = 3\sin^2(4x+1)\cos(4x+1)\cdot 4 = 12\sin^2(4x+1)\cos(4x+1)$

b) $g(x) = \int_1^x (1+t^2)t\,dt$

Answer. By the fundamental theorem of the calculus, $g'(x) = (1+x^2)x$.

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3. Integrate:

a) $\int (x^2+1)^2 x\,dx$

Answer. Let $u = x^2+1,\ du = 2x\,dx$:

$$\int (x^2+1)^2 x\,dx = \frac{1}{2}\int u^2 du = \frac{1}{2}\frac{1}{3}(x^2+1)^3 + C = \frac{1}{6}(x^2+1)^3 + C$$

b) $\int \tan x \sec^2 x\,dx$

Answer. Let $u = \tan x,\ du = \sec^2 x\,dx$:

$$\int \tan x \sec^2 x\,dx = \int u\,du = \frac{1}{2}\tan^2 x + C$$

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4. Integrate:

a) $\int_1^4 \dfrac{1}{\sqrt{y}(\sqrt{y}+1)^2}dy$

Answer. Letting $u = y^{1/2}+1,\ \ du = (1/2)y^{-(1/2)}dy$, we get:

$$= \frac{1}{2}\int_2^3 \frac{du}{u^2} = -\frac{1}{2}u^{-1}\Big|_2^3 = -\frac{1}{2}[\frac{1}{3}-\frac{1}{2}] = \frac{1}{12}$$

b) $\int_0^{\pi/2} \cos^2 x \sin x dx$

Answer. Let $u = \cos x,\ du = -\sin x dx$. When $x = 0,\ u = 1$ and when $x = \pi/2,\ u = 0$. We get

$$= -\int_1^0 u^2 du = -\frac{u^3}{3}\Big|_1^0 = \frac{1}{3}$$

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5. Find the slope of the tangent line to the curve $\cos x + \sin y = 3/2$ at the point $(\pi/3, \pi/2)$.

Answer. Differentiate implicitly:

$$-\sin x + \cos y \frac{dy}{dx} = 0.$$

Now evaluate at $(\pi/3, \pi/2)$ and solve for dy/dx:

$$-\frac{\sqrt{3}}{2} + \frac{dy}{dx} = 0, \quad \text{so that} \quad \frac{dy}{dx} = \frac{\sqrt{3}}{2} .$$

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6. A conical water tank of height 8 ft, base radius 5 ft, stands on its vertex. Water is flowing in at the top at a rate of 2.5 ft^3/min. At what rate is the water level rising when that level is at 3 ft? The volume of a cone of base radius r and height h is $(1/3)\pi r^2 h$.

Answer. Let x be height of the water, and r the radius of the surface of water at time t. Then, by similar triangles,

$$\frac{x}{8} = \frac{r}{5} , \quad \text{so} \quad r = \frac{5}{8} x .$$

Thus the volume and the water height are related by

$$V = \frac{1}{3}\pi r^2 x = \frac{25\pi}{3 \cdot 64} x^3 .$$

Differentiate and set $x = 3$, $dv/dt = 2.5$:

$$\frac{dV}{dt} = \frac{25\pi}{3 \cdot 64} 3x^2 \frac{dx}{dt} , \quad \text{so} \quad 2.5 = \frac{25\pi}{3 \cdot 64} 3(3)^2 \frac{dx}{dt} ,$$

from which we conclude

$$\frac{dx}{dt} = \frac{2.5 \cdot 3 \cdot 64}{25\pi(27)} = \frac{196}{270\pi} .$$

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7. A farmer wishes to enclose a rectangular field of 1,000 square yards so that one side is brick and the other three sides are chain link fence. A Brick wall costs \$18 a linear yard and chain link, \$ 6 a linear yard. Find the dimensions of the field which minimizes the cost.

Answer. Let x be the length of the side of brick, and y the length of the other side. Then

$$A = 1000 = xy , \quad \text{so} \quad y = 1000x^{-1} ,$$

$$C = 18x + 6x + 6 \cdot 2y = 24x + 12000x^{-1} ,$$

$$C' = 24 - 12000x^{-2} ,$$

$$24x^2 = 12000, \quad x^2 = 500, \quad x = 10\sqrt{5} \quad \text{and} \quad y = 20\sqrt{5} .$$

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8. Find the solution to the differential equation

$$\frac{dy}{dx} = y^2x^2 + y^2$$

such that $y(1) = 2$.

The variables separate;

$$y^{-2}dy = (x^2+1)dx\ ,$$

so we can integrate both differentials, obtaining

$$-y^{-1} = \frac{x^3}{3} + x + C.$$

Now, evaluate at $x = 1,\ y = 2$:

$$-\frac{1}{2} = \frac{1}{3} + 1 + C$$

so $C = -11/6$ and

$$y = \frac{1}{11/6 - \frac{x^3}{3} - x}$$

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9. Graph

$$y = \frac{x^3}{x^2-1}$$

showing clearly all asymptotes and local maxima and minima.

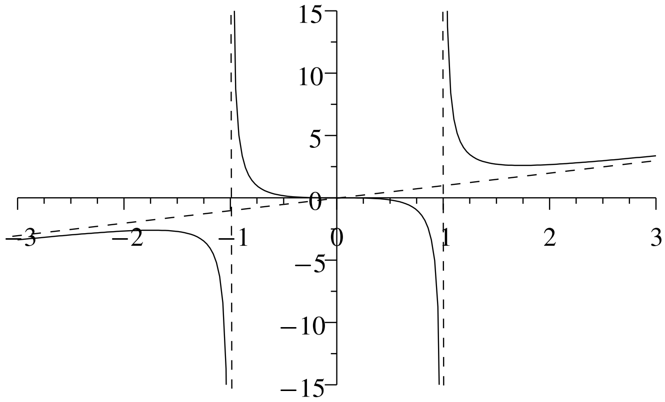
Answer. The vertical asymptotes are at $x = \pm 1$, and the horizontal asymptote is (by long division): $y = x$. Now, the numerator is negative for x negative and positive for x positive, and the denominator is negative for $|x| < 1$ and otherwise positive. Calculating the derivative, we find

$$\frac{dy}{dx} = \frac{(x^2-1)(3x^2) - x^3(2x)}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2}\ .$$

We make the table of values in the relevant intervals:

x	< -1	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$> \sqrt{3}$
y	*neg*	*neg*	*pos*	*neg*	*pos*	*pos*
y'	*pos*	*neg*	*neg*	*neg*	*neg*	*pos*

Using this information we get the graph



10. What is the area of the region bounded by the curves $y = x^3 - 3x$ and $y = 3x$.

Answer. First find the points of intersection:

$$x^3 - x = 3x, \qquad x^3 = 4x$$

has the solutions $x = 0, 2$. The line $y = 3x$ lies above the curve $y = x^3 - x$. Thus, the area is:

$$\int_0^2 [3x - (x^3 - x)]dx = \int_0^2 (4x - x^3)dx = 2x^2 - \frac{x^4}{4}\Big|_0^2 = 8 - 16/4 = 4.$$

11. The region in the first quadrant under the curve $y^2 = 2x - x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Answer. We use the disc method; here $dV = \pi y^2 dx$, so

$$V = \int_0^2 \pi(2x - x^2)dx = \frac{4\pi}{3}$$

12. The region between the curves $y = 8x$ and $y = x^4$ is rotated about the y-axis. Find the volume of the resulting solid.

Answer. Using the methods of shells, we have

$$dV = 2\pi x(8x - x^4)dx$$

The curves intersect at the points (0,0) and (2,16). Thus

$$V = \int_0^2 2\pi(8x^2 - x^5)dx = 2\pi(\frac{8}{3}x^3 - \frac{x^6}{6})\Big|_0^2 = \frac{2^6\pi}{3}\ .$$

This can also be done by the methods of washers, integrating in the y-direction from $y = 0$ to $y = 16$.

13. Find the length of the curve $y = t^3$, $x = t^2$, $0 \le t \le 1$.

Answer. The basic equation for arc length is $ds^2 = dx^2 + dy^2$. Here $dy = 3t^2dt$, $dx = 2tdt$, so $ds^2 = (4t^2 + 9t^4)dt^2$, and thus

$$ds = t\sqrt{4 + 9t^2}$$

$$L = \int_0^1 t\sqrt{4 + 9t^2}dt = \frac{1}{18}\int_4^{13} u^{1/2}du$$

which comes out to $(1/27)[13^{3/2} - 8]$.

14. Find the work done in pumping all the oil (whose density is 50 lbs. per cubic foot) over the edge of a cylindrical tank which stands on end. Assume that the radius of the base is 4 feet, the height is 10 feet and the tank is full of oil.

Answer. The slab of oil of thickness dh at a depth h has to be lifted a height h. The work to do this is $dW = 50(\pi 4^2)dh \cdot h$. Thus

$$W = \int_0^{10} 800\pi h dh = 800\pi\frac{h^2}{2}\Big|_0^{10} = 40,000\pi$$

foot-pounds.

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15. Find the center of mass of the homogeneous region in the first quadrant bounded by the curve $x^4 + y = 1$.

Answer. The region is given by $0 \le y \le 1 - x^4,\ 0 \le x \le 1$. Its mass is

$$\int_0^1 (1 - x^4)dx = (x - \frac{x^5}{5})|_0^1 = 4/5.$$

The moment about the y-axis is

$$\int_0^1 x(1 - x^4)dx = (\frac{x^2}{2} - \frac{x^6}{6})|_0^1 = 1/3.$$

To find the moment about the x-axis we can change coordinates, so that we sweep out the area in the y direction. Then, we write the region as: $0 \le x \le (1 - y)^{1/4},\ 0 \le y \le 1$. Thus the moment about the x-axis is

$$\int_0^1 y(1 - y)^{\frac{1}{4}} dy$$

which we integrate by the substitution $u = 1 - y$. When $y = 0,\ u = 1$ and when $y = 1,\ u = 0$, so the moment is

$$= -\int_1^0 (1 - u)u^{\frac{1}{4}} du = -\int_1^0 (u^{\frac{1}{4}} - u^{\frac{5}{4}})du = 16/45$$

Thus the center of mass has the coordinates $(\frac{1}{3}/\frac{4}{5}, \frac{16}{45}/\frac{4}{5}) = (\frac{5}{12}, \frac{4}{9})$.