Calculus I
Practice Exam 2 Answers, Summer 2002

1. Find the equation of the tangent line to the ellipse

$$x^2 + 2y^2 = 9$$

at the point (1,-2).

Answer. Take the differential: $2xdx + 4ydy = 0$. Now put in the coordinates of the point: $2(1)dx + 4(-2)dy = 0$. This is the relation between increments of the variables along the tangent line. But these increments, measured from (1,-2) are $x-1$, $y-(-2)$. Thus the equation of the tangent line is

$$2(x-1) - 8(y+2) = 0 \ , \quad \text{or} \quad 2x - 8y = 18 \ .$$

---

2. Given the relation

$$x^3 - y^2 + x + 2y = 7$$

between the variables x and y, express the derivative dy/dx as a function of x and y.

Answer. Take the differential and then solve for dy/dx:

$$3x^2dx - 2y^2dy + dx + 2dy = 0$$

$$(3x^2+1)dx + (2-2y^2)dy = 0$$

$$\frac{dy}{dx} = \frac{3x^2+1}{2(y^2-1)} \ .$$

---

3. A curve C in the plane is the graph of the relation $\sqrt{y} = \sin(x/2)$.
a) Find the slope of the tangent line to the curve at the point $(\pi/2, 1/2)$.

Answer. Differentiate:

$$\frac{1}{2}y^{-1/2}\frac{dy}{dx} = \frac{1}{2}\cos(x/2) \ , \quad \text{so that} \quad \frac{dy}{dx} = y^{1/2}\cos(x/2) \ .$$

Evaluating at $(\pi/2, 1/2)$, the slope is

$$\sqrt{\frac{1}{2}}\cos(\pi/4) = \frac{1}{2} \ .$$

b) Find a point in the first quadrant where the tangent line is horizontal.

Answer. Any point where $y^{1/2}\cos(x/2) = 0$ will do. For $x = \pi$, we have $\cos(\pi/2) = 0$, so that will do. The corresponding y value satisfies $\sqrt{y} = \sin(\pi/2) = 1$: thus $(\pi, 1)$ is such a point.

---

4. Find those points on the curve $x^2 - 2xy + 3y^2 = 294$ at which the tangent line is vertical.

Answer. Take the differential to see the relation between the increments along the tangent line:

$$2xdx - 2ydx - 2xdy + 6ydy = 0 \ .$$

Since the increment dx along a vertical line is 0, this gives as the condition: $-2x+6y=0$, or $x=3y$. Substituting this in the equation of the curve gives $9y^2-6y^2+3y^2=294$, or $6y^2=294$, so $y=\pm 7$. The desired points are then $\pm(21,7)$.

---

5. At a certain moment the hare and the mouse start out from the same point; the hare moving east at 8 mph, and the mouse north at 5 mph. After 3 hours, at what rate is the distance between them increasing?

Answer. Let h be the distance travelled by the hare, m, the distance travelled by the mouse, and x, the distance between them. Then $x^2=h^2+m^2$, so (differentiating with respect to time):

$$x\frac{dx}{dt}=h\frac{dh}{dt}+m\frac{dm}{dt}\ .$$

Now $dh/dt=8$ and $dm/dt=5$, so after 3 hours, $h=24$, $m=15$ and $x=\sqrt{(24)^2+(15)^2}=28.3$. Putting these values in the above equation, we have:

$$28.3\frac{dx}{dt}=24\cdot 8+15\cdot 3=237$$

so $dx/dt=237/28.3=8.37$ mph.

---

6. At a race, I am sitting along the finish line, exactly 100 feet away from the track. The lead car is travelling at 560 feet/sec. Asuming I am watching the lead car, at what rate is my angle of vision changing as it crosses the finish line?

Answer. Let θ represent the angle of vision between the position of the car and the finish line, and x the distance of the car from the finish line. The $\tan\theta=x/100$. Differentiating: $\sec^2\theta(d\theta/dt)=(1/100)(dx/dt)$. Now at the moment the car crosses the finish line, $\theta=0$ and $\sec\theta=1$. We are given $dx/dt=560$, so

$$(1)^2\frac{d\theta}{dt}=\frac{560}{100}=5.6\ \text{rad/sec}\ .$$

---

7. Water is pouring into a cone-shaped container (with vertex pointing down) at a rate of .05 cubic feet per minute. The cone is 8 feet deep and has a 5 foot diameter at the top. How fast is the water level rising when it is 3 ft. high?

Answer. The volume of a cone of base radius r and height h is $V=(1/3)\pi r^2h$. By similar triangles (draw a picture!), when the water is h feet high up the cone, we have $h/8=r/2.5$, so $r=(2.5/8)h=.3125h$. Thus, in terms of h alone,

$$V=\frac{1}{3}\pi(.3125h)^2h=.92h^3\ .$$

Differentiating with repsect to t:

$$\frac{dV}{dt}=.92(3h^2)\frac{dh}{dt}\ .$$

We are given that $dV/dt=.05$ ft^3/min. Thus, at $h=3$,

$$\frac{dh}{dt}=\frac{.05}{.92(3)(3)^2}=.002\ \text{ft/min}\ .$$

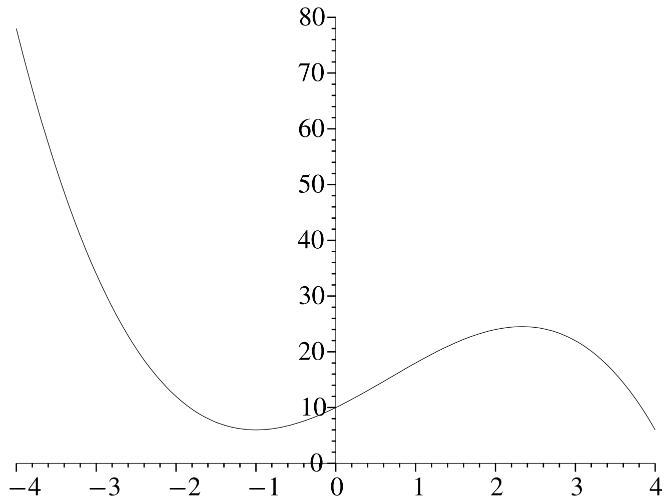
---

8. Let $y=10+7x+2x^2-x^3$. For what value of x is y a local maximum? a local minimum?

Answer. Differentiate: $y' = 7 + 4x - 3x^2$. The critical points are the roots; by the quadratic formula these are

$$x = \frac{-4 \pm \sqrt{16+84}}{-6} = -1 \ , \ \frac{7}{3} \ .$$

Now $y'' = 4 - 6x$, so is positive at $x = -1$, and thus is the local minimum; 7/3 is the local maximum. (See the the figure for verification.)



---

9. I want to build a rectangular building on a square base which has 60000 ft^3 of space. My painter will charge $\$h/2$ per ft^2 to paint a building h feet high, and will paint the roof at a cost of \$30 per ft^2. How high should the building be to minimize the cost of painting?

Answer. Let b be the length of a side of the base, and h the height of the building. Then the volume of the building is $b^2h = 60000$ ft^3. The cost of painting the building (there are four sides of bh ft^2 each, and the roof has b^2 ft^2):

$$C = \frac{h}{2}(4bh) + 30b^2 = 2bh^2 + 30b^2 \ .$$

Now, since the constraint is $b^2h = 60000$, and we want to solve for the height, we substitute $b^2 = 6 \times 10^4 h^{-1}$, $b = 100\sqrt{6}h^{-1/2}$ in this equation, obtaining

$$C = 200\sqrt{6}h^{3/2} + (30)(6 \times 10^4)h^{-1} \ .$$

To mimimize C, we find where the derivative is zero:

$$C' = 300\sqrt{6}h^{1/2} - 18 \times 10^5 h^{-2}$$

and solve

$$300\sqrt{6}h^{1/2} = 18 \times 10^5 h^{-2}$$

$$h^{5/2} = \frac{18 \times 10^5}{300\sqrt{6}} = \sqrt{6} \times 10^3 \ ,$$

so $h = 22.68$ ft. (and $b = 51.43$ ft.).

---

10. I have an elliptical meadow whose major axis is 100 meters and whose minor axis is 60 meters. I want to put a rectangular soccer field in the meadow, two of whose sides are parallel to the major axis, What are the dimensions of the field which maximize its area?

Answer. We can model the meadow as in the figure. The equation defining the boundary of the meadow is

$$\left(\frac{x}{50}\right)^2 + \left(\frac{y}{30}\right)^2 = 1 \quad \text{or} \quad 900x^2 + 2500y^2 = 225 \times 10^4 \ .$$

The area of the soccer field is $A = 4xy$; we want to find x and y which maximize A. First, solve the equation of the ellipse for y in terms of x, and sbustitute in the formula for A, obtaining:

$$A = 4x\frac{(225 \times 10^4 - 900x^2)^{1/2}}{50} \ .$$

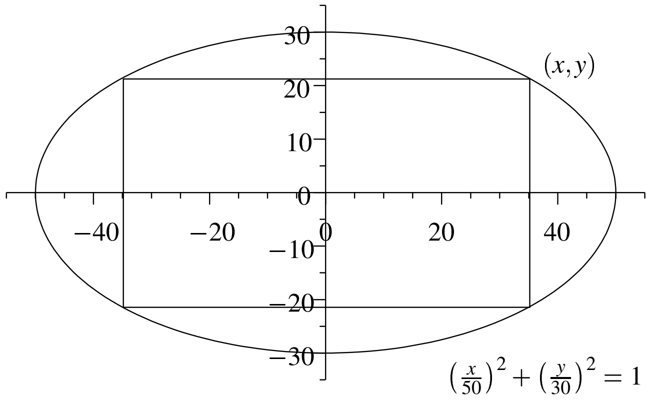
Now, differentiate:

$$A' = \frac{4}{50}\left[((225 \times 10^4 - 900x^2)^{1/2} + x\frac{1}{2}(225 \times 10^4 - 900x^2)^{-1/2}(-1800x)\right] \ .$$

Solve for $A' = 0$:

$$225 \times 10^4 - 900x^2 = 900x^2$$

which gives $x = 35.3$ meters and $y = 21.2$ meters, so the soccer field is 70.6 m. by 42.4 m.



---

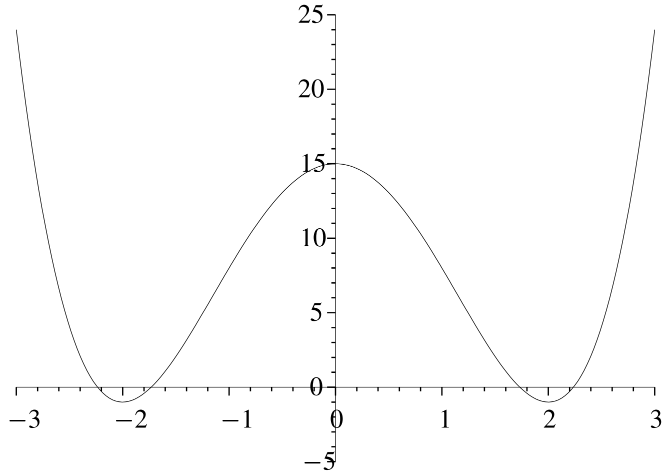
11. Let $y = (x^2 - 3)(x^2 - 5)$. Sketch the graph, clearly showing all local maxima and minima, points of inflection and concavity.

Answer. Differentiate twice:

$$y' = (x^2 - 3)(2x) + (x^2 - 5)(2x) = 4x(x^2 - 4) \ ,$$

$$y'' = 4(x^2 - 4) + 4x(2x) = 12x^2 - 16 \ .$$

$y' = 0$ when $x = 0, \pm 2$; these are the critical points. $y'' = 0$ when $x = \pm\sqrt{4/3}$; these are the points of inflection. Checking the sign of y'' we see that the graph is concave down when $-\sqrt{4/3} < x < \sqrt{4/3}$, and otherwise concave up. In particular, f'' is positive at the critical points ± 2, so these are local minima, and $x = 0$ is a local maximum. See the figure.



12. Graph

$$y = \frac{x^3}{x^2 - 1}$$

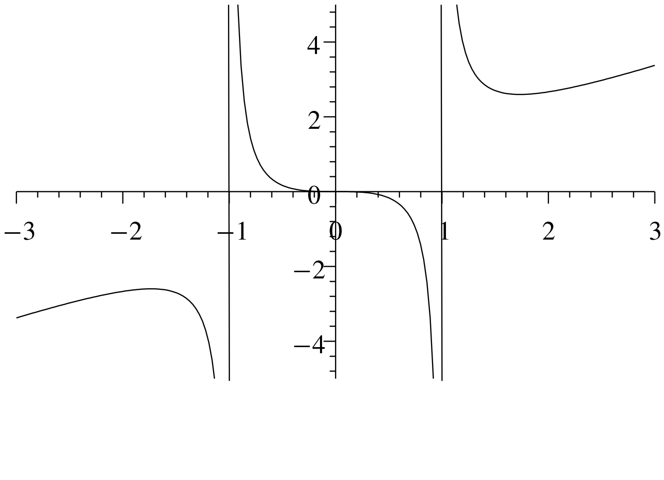
Answer. By long division,

$$y = \frac{x^3}{x^2 - 1} = x + \frac{x}{x^2 - 1}$$

so the horizontal asymptote is $y = x$. Notice that for x very negative, the remainder is negative, so the graph is under the asymptote to the left, and similarly, above the asymptote to the right. The vertical asymptotes are at $x = \pm 1$. If x is slightly more negative than -1, y is negative. Thus $y \to -\infty$ as $x \to -1$ from the left. However, the sign changes to the right of -1, so $y \to \infty$ as $x \to -1$ from the right. The same is true at $x = 1$. Thus the graph should look like that in PE2.12.
We locate the local maximum and minimum by taking the derivative:

$$y' = \frac{(x^2 - 1)(3x^2) - x^3(2x)}{(x^2 - 1)^2} = \frac{x^2(x^2 - 3)}{(x^2 - 1)^2}$$

which is zero when $x = 0, \pm\sqrt{3}$. The local maximum is at $-\sqrt{3}$, the local minimum at $\sqrt{3}$, and the curve is very flat at $x = 0$.

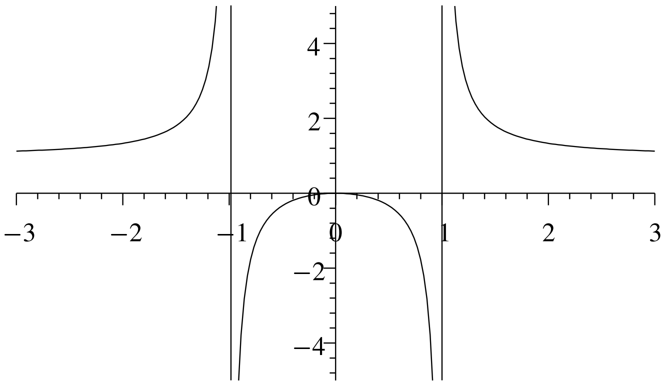


13. Graph

$$y = \frac{x^2}{x^2 - 1}$$

Answer. Since the numerator and denominator have the same degree, the horizontal asymptote is $y = 1$. For $|x| > 1$ the numerator is larger than the denominator, so the curve is always above its horizontal asymptote. The vertical asymptotes are $x = \pm 1$. Since $y > 1$ for $|x| > 1$, the curve has to go to $+\infty$ both on the left of -1 and on the right of 1. For $|x| < 1$, the value is negative, so the curve goes to $-\infty$ on the insides of the asymptotes. This suggests the figure. Differentiating, we confirm that the curve is always increasing for $x < 0$ and decreasing for $x > 0$:

$$y' = \frac{(x^2 - 1)(2x) - x^2(2x)}{(x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2} .$$



14. Graph

$$y = \frac{x}{x^2 - 1}$$

Answer. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$. The rest of the analysis is as in problems 12 and 13 leads to the graph in PE2.14.