Calculus I
Practice Exam 1, Summer 2002, Answers

1. Find the equation of the line which goes through the point (2,-1) and is perpendicular to the line given by the equation $2x - y = 1$.

Answer. The given line has slope 2, so the line we seek has slope -1/2. (2,-1) lies on the line, so the equation is

$$\frac{y-(-1)}{x-2} = -\frac{1}{2}$$

which simpifies to $y = -(1/2)x$.

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2. a) Let $f(x) = x^2 + 3x - 1$. Find the slope of the line joining the points $(2,9)$ and $(x, f(x))$.
b) Find the slope of the tangent line to the curve $y = f(x)$ at the point $(2,9)$.
c) What is the equation of this tangent line?
Answer. a) The line joining these two points has slope

$$\frac{x^2+3x-1-9}{x-2} = \frac{x^2+3x-10}{x-2} = x+5$$

by long division.
b). $f'(x) = 2x+3$, so at $x = 2$, the tangent line has slope $f'(2) = 7$. (Notice that if we substitute $x = 2$ in the answer to part a), we get the same answer, confirming the statement after equation (1.11)).

$$\frac{y-9}{x-2} = 7 \qquad \text{or} \qquad y = 7x - 5 \, .$$

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3. Let $y = x^3 - 3x + 1$. Find the points on the curve whose tangent lines have slope $m = 9$.

Answer. The slope of the tangent line at (x, y) is $dy/dx = 3x^2 - 3$. Set this equal to 9 and solve:

$$3x^2 - 3 = 9 \qquad \text{or} \qquad 3x^2 = 12$$

which has the solutions $x = \pm 2$.

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4. Find the derivatives of the following functions:
a) $f(x) = x^3 - x^2 + 1$
b) $g(x) = x^2 + \dfrac{1}{x^3}$
c) $h(x) = (x^2 + \dfrac{1}{x^3})(x^3 - x^2 + 1)$
Answer. a) $f'(x) = 3x^2 - 2x$.

b) First write the function in exponential notation: $g(x) = x^2 + x^{-3}$. Then $g'(x) = 2x - 3x^{-4}$.

c) We use the product rule and the answers to parts a) and b):

$$h'(x) = (2x - 3x^{-4})(x^3 - x^2 + 1) + (x^2 + \frac{1}{x^3})(3x^2 - 2x)$$

5. Find the derivatives of the given functions:
a) $f(x) = 3x^{-1} + x^3$
b) $g(x) = (x^3+1)^4$
c) $h(x) = (\cos(2x)+1)\sin(3x)$

Answer. a) $f'(x) = -3x^{-2} + 3x^2$.

b) $g'(x) = 4(x^3+1)^3(3x^2) = 12x^2(x^3+1)^3$.

c) Use the product rule and chain rules carefully:

$$h'(x) = (\cos(2x)+1)(\cos(3x)(3) + (-\sin(2x)(2)(\sin(3x))$$

$$= 3(\cos(2x)+1)\cos(3x) - 2\sin(2x)\sin(3x)$$

6. Find the derivative of

$$f(x) = \frac{x^2+1}{x+1}$$

Answer.

7. Find the derivatives of the following functions:
a) $f(x) = \cos^2 x$
b) $g(x) = \frac{\sin x}{\cos^2 x}$

Answer. a) $f'(x) = 2\cos x(-\sin x) = -2\cos x \sin x$.

$$g'(x) = \frac{\cos^2 x \cos x - \sin x(-2\cos x \sin x)}{\cos^4 x} = \frac{\cos^3 x + 2\cos x \sin^2 x}{\cos^4 x}$$

$$= \frac{\cos^2 x + 2\sin^2 x}{\cos^3 x} = \frac{1+\sin^2 x}{\cos^3 x}$$

which is where you would get if you first wrote $g(x) = \tan x \sec x$, and then used the product rule.

8. Find the equation of the line tangent to the curve $y = \cos(x/2)$ at $(3\pi, 0)$

Answer. Take differentials: $dy = -(1/2)\sin(x/2)dx$. At $x = 3\pi$, we get $dy = -(1/2)(-1)dx = dx/2$. Thus the equation of the tangent line is $y = (x-3\pi)/2$.

9. Let $f(x) = x^3 - 8x^2 + 3$. Find the interval in which $f'(x) < 0$.

Answer. $f'(x) = 3x^2 - 16x = x(3x-16)$. This is zero when $x = 0$ and $x = 16/3$. f' is of constant sign in the intervals separated by these points, so we need only check for a particular point. Since $f'(1) = -13$, $f'(x) < 0$ for x in $(0, 16/3)$.

10. An object moves in a straight line so that its position at time t is given by $x(t) = t(t^2+1)^2$. What is the velocity of the object when $t = 2$?

Answer.

$$v = \frac{dx}{dt} = (t^2+1)^2 + t(2(t^2+1))(2t) = (t^2+1)(t^2+1+4t^2) = (t^2+1)(5t^2+1)\ .$$

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11. Let $f(x) = (x - \sqrt{x})^2$. Find $f'(x)$ and $f''(x)$.

Answer.

$$f(x) = (x-\sqrt{x})^2 = x^2 - 2x\sqrt{x} + x = x^2 - 2x^{3/2} + x$$

$$f'(x) = 2x - 3x^{1/2} + 1 \qquad f''(x) = 2 - \frac{3}{2}x^{-1/2}\ .$$

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12. Sketch the graph of a function with these properties:
a) $f(0) = 2$ and $f(1) = 0$;
b) $f'(x) < 0$ for $0 < x < 2$;
c) $f'(x) > 0$ for $x < 0$ or $x > 2$.

Answer.