Mathematics 1210-90 Final Examination. July 29,30, 2003

1. Find the derivatives of the following functions:

a) $$f(x) = (x^2 - 3)^2(2x + 4)$$

Answer. $$f'(x) = 2(x^2 - 3)(2x)(2x + 4) + (x^2 - 3)^2(2) = 2(x^2 - 3)(5x^2 + 8x - 3)\ .$$

b) $$g(x) = \frac{\sin x}{\cos x + 1}$$

Answer.
$$\begin{aligned} g'(x) &= \frac{(\cos x + 1)\cos x - \sin x(-\sin x)}{(\cos x + 1)^2} \\ &= \frac{1}{\cos x + 1} \end{aligned}$$

2 *a*) Integrate : $$\int (6x^2 + 1)^5 x dx$$

Answer. Let $u = 6x^2 + 1,\ du = 12xdx$. Then

$$\int (6x^2 + 1)^5 x dx = \frac{1}{12}\int u^5 du = \frac{1}{72}(6x^2 + 1)^6 + C\ .$$

2 *b*) Integrate : $$\int_1^4 \frac{1}{\sqrt{y}(\sqrt{y} + 1)^2} dy$$

Answer. Let $u = y^{1/2},\quad du = (1/2)y^{-(1/2)}dy$,

$$= 2\int_1^2 \frac{du}{(u + 1)^2} = -2(u + 1)^{-1}|_1^2 = -2[\frac{1}{3} - \frac{1}{2}] = \frac{1}{3}$$

3. Find the equation of the tangent line to the curve $2x^2 + y^2 - 2xy = 17$ at the point (1,5).

Answer. Take the differential: $4xdx + 2ydy - 2ydx - 2xdy = 0$. Now evaluate at the point (1,5), to obtain

$$4dx + 10dy - 10dx - 2dy = 0\ , \quad \text{which becomes} \quad -6dx + 8dy = 0\ .$$

Now replace the differentials by the increments:

$$-6(x-1)+8(y-5)=0 \quad \text{which simplifies to} \quad -6x+8y=34 \ .$$

4. Consider the region in the first quadrant bounded by the curve $y = 12 - \frac{3}{4}x^2$. What are the dimensions of the largest rectangle with sides parallel to the coordinate axes which can be inscribed inside this region?

Answer. Let (x, y) be the dimensions of the rectangle,so that the area is $A = xy$. Since this point must be on the boundary of the region, its coordinates satisfy the equation: $y = 12 - (3/4)x^2$. Thus, we have

$$A = x(12 - \frac{3}{4}x^2) = 12x - \frac{3}{4}x^3 \ .$$

To find the maximum, differentiate and set equal to zero:

$$\frac{dA}{dx} = 12 - \frac{9}{4}x^2 = 0 \ ,$$

which has the solution $x = 4\sqrt{3}/3$. To find y put this in the original equation, getting $y = 8$. Thus the answer is $(4\sqrt{3}/3, 8)$.

5. Sketch the graph of the function $y = 3x^4 - 4x^3 - 12x^2 + 2$. Find the x values of all local minima, maxima and points of inflection.

Answer. First, differentiate:

$$y' = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1) \ .$$

Thus the critical points are at $x = -1, 0, 2$. Since the function is of fourth degree, it approaches $+\infty$ as x leaves the paper to the left or to the right. Thus, we expect $x = -1$ to be a local minimum, $x = 0$ a local maximum and $x = 2$ a local minimum. Taking the second derivative;

$$y'' = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2) \ .$$

Evaluating at the critical points, we verify that $y'' > 0$ at $x = -1, 2$, and $y'' < 0$ at $x = 0$, confirming our expectations. Using the quadratic formula, the roots are $x = (1 \pm \sqrt{7})/3$; these are the points of inflection. The curve is concave up until $x = (1 - \sqrt{7})/3$, then concave down, and finally concave up to the right of $x = (1 + \sqrt{7})/3$.

6. Find the solution to the differential equation

$$\frac{dy}{dx} = y^2x^2 + y^2$$

such that $y(1) = 2$.

Answer. We separate variables to put the equation in the form $y^{-2}dy = (x^2+1)dx$. Now we integrate both sides:

$$-y^{-1} = \frac{x^3}{3} + x + C.$$

To solve for C, put in the initial conditions:

$$-\frac{1}{2} = \frac{1}{3} + 1 + C$$

so $C = -11/6$ and

$$y = \frac{-1}{\frac{x^3}{3} - x - \frac{11}{6}}$$

7. Find the area between the curve

$$y = \frac{x+1}{x^3}$$

and the x-axis, as x ranges from 1 to 4.

Answer. The area is

$$Area = \int_1^4 \frac{x+1}{x^3}dx = \int_1^4 (x^{-2} + x^{-3})dx = (-x^{-1} - \frac{1}{2}x^{-2})\Big|_1^4 = \frac{39}{32}$$

8. The region between the curves $y = 8x$ and $y = x^4$ is rotated about the y-axis. Find the volume of the resulting solid.

Answer. Using the methods of shells, we have

$$dV = 2\pi x(8x - x^4)dx$$

The curves intersect at the points (0,0) and (2,16). Thus

$$V = \int_0^2 2\pi(8x^2 - x^5)dx = 2\pi(\frac{8}{3}x^3 - \frac{x^6}{6})\Big|_0^2 = \frac{2^6\pi}{3} = \frac{64\pi}{3} = 67.0206\ .$$

This can also be done by the methods of washers, integrating in the y-direction from $y = 0$ to $y = 16$. Now, $dV = \pi((y^{1/4})^2 - (y/8)^2)dy$, so

$$V = \int_0^{16} \pi(y^{1/2} - \frac{y^2}{64})dy = \pi(\frac{2}{3}y^{3/2} - \frac{y^3}{3(64)}\Big|_0^{16} = \frac{2\pi}{3}(2(64) - 64) = 67.0206$$

9. Consider a curve given parametrically by $x = 4\cos^2 t,\ y = 3\sin^2 t$. Find the length of the piece of this curve running from $t = 0$ to $t = \pi/2$.

Answer. To calculate the element of arc length, ds, we first take differentials:

$$dx = -8\cos t \sin t dt \ , \quad dy = 6\sin t \cos t dt \ ,$$

so that

$$ds^2 = (64\cos^2 t \sin^2 t + 36\cos^2 t \sin^2 t)dt^2 \quad \text{and} \quad ds = 10\cos t \sin t dt \ .$$

Thus

$$Length = 10\int_0^{\pi/2} \cos t \sin t dt = 10\frac{\sin^2 t}{2}\Big|_0^{\pi/2} = 5 \ .$$

10. FInd the center of mass of the homogeneous region in the first quadrant bounded by the curve $x^4 + y = 1$.

Answer. The region is given by $0 \le y \le 1 - x^4,\ 0 \le x \le 1$. Its mass is

$$\int_0^1 (1 - x^4)dx = (x - \frac{x^5}{5})|_0^1 = 4/5.$$

The moment about the y-axis is

$$\int_0^1 x(1 - x^4)dx = (\frac{x^2}{2} - \frac{x^6}{6})|_0^1 = 1/3.$$

The moment about the x-axis is

$$\int_0^1 y(1-y)^{\frac{1}{4}} dy$$

which we integrate by the substitution $u = 1 - y$,

$$= \int_0^1 (1-u)u^{\frac{1}{4}} du = \int_0^1 (u^{\frac{1}{4}} - u^{\frac{5}{4}})du = 16/45$$

Thus the center of mass has the coordinates $(\frac{1}{3}/\frac{4}{5}, \frac{16}{45}/\frac{4}{5}) = (\frac{5}{12}, \frac{4}{9})$.