Calculus I
Final Exam, Spring 2003, Answers

1. Find the derivatives of the following functions:

a) $f(x) = \dfrac{\sin x}{\sin x + 1}$

Answer. $f'(x) = \dfrac{(\sin x+1)(\cos x) - (\sin x)(\cos x)}{(\sin x+1)^2} = \dfrac{\cos x}{(\sin x+1)^2}$

b) $g(x) = x\sqrt{1+x^2}$

Answer. Write $g(x) = x(1+x^2)^{1/2}$ and use the product rule:

$$g'(x) = (1)(1+x^2)^{1/2} + x\frac{1}{2}(1+x^2)^{-1/2}(2x) = \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}} = \frac{1+2x^2}{\sqrt{1+x^2}}$$

---

2. Integrate:

a) $\int x\sqrt{1+x^2}dx =$

Answer. Let $u = 1+x^2,\ du = 2xdx$. Then

$$\int x\sqrt{1+x^2}dx = \frac{1}{2}\int u^{1/2}du = \frac{1}{2}\frac{2}{3}u^{3/2} + C = \frac{1}{3}(1+x^2)^{3/2} + C\ .$$

b) $\int \tan x \sec^2 x dx =$

Answer. Let $u = \tan x,\ du = \sec^2 x dx$. Then

$$\int \tan x \sec^2 x dx = \int u du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C\ .$$

---

3. A trough 12 feet long has a triangular cross section. The triangle is 4 feet across at the top and 3 feet deep. Water is pouring into the trough at the rate of 4 ft^3/min. At what rate is the height of the water rising when it is 2 feet above the vertex of the triangle?

Answer. Let h be the height of the water, and x the width of the top of the water. By similar triangles,

$$\frac{h}{3} = \frac{x}{4} \quad \text{so that} \quad x = \frac{4}{3}h\ .$$

Let V the volume of water in the trough. Then

$$V = \frac{1}{2}xh(12) = \frac{1}{2}\frac{4}{3}(h)(h)(12) = 8h^2\ . \tag{1}$$

We are told that $dV/dt = 4$ ft^3/min. Differentiating (1), we get

$$\frac{dV}{dt} = 16h\frac{dh}{dt}\ , \quad \text{so that} \quad 4 = 16(2)\frac{dh}{dt}\ ,$$

when $h = 2$. This gives us $dh/dt = 1/8 = .125$ ft/min.

4. A plane heading north at 480 mph passes over Salt Lake City at 1:00pm. Another plane, heading east at 360 mph passes over Salt Lake City at 1:15 pm. At what time are the planes closest to each other?

Answer. Let $N(t)$ be the distance from Salt Lake of the northbound plane at t hours after 1:00 pm, and $E(t)$ that of the eastbound plane at that time. Then

$$N(t) = 480t \ , \quad \text{and} \quad E(t) = 360(t - .25) \ ,$$

since the eastbound plane is directly above Salt Lake a quarter hour after the northbound. The distance between the planes is $\sqrt{N^2+E^2}$, so we solve our problem by finding the minimum of

$$f(t) = N^2 + E^2 = (480t)^2 + (360(t - .25))^2 \ .$$

Differentiating,

$$f'(t) = 2(480)(480t) + 2(360)((360(t - .25)) \ .$$

Setting this equal to zero, we get

$$(480^2 + 360^2)t = 360^2(.25) \quad \text{or} \quad t = \frac{360^2(.25)}{480^2+360^2} = .09 \text{ hr} = 5.4 \text{ min}$$

after 1;00 pm, so at 1:05:24 pm they are closest.

5. Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{3x+1}{y}$$

such that $y(0) = 4$.

Answer. Separating variables we obtain $ydy = (3x+1)dx$. Integrating:

$$\frac{y^2}{2} = \frac{(3x+1)^2}{6} + C \ , \quad \text{or} \quad y^2 = \frac{1}{3}(3x+1)^2 + C \ .$$

The initial conditions give us $4^2 = 1/3 + C$, so $C = 47/3$. Thus

$$y = \sqrt{\frac{(3x+1)^2 + 47}{3}} \ .$$

We could also have integrated the differential equation to

$$\frac{y^2}{2} = \frac{3}{2}x^2 + x + C \ ;$$

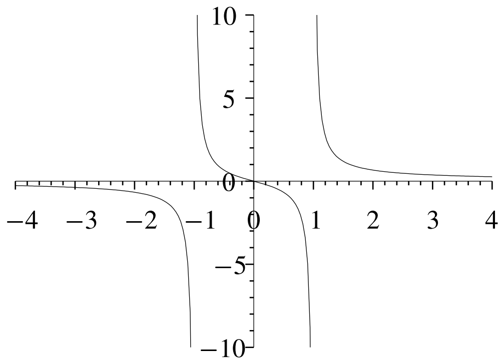
found this C to be $C = 8$, and obtained the answer (which is the same)

$$y = \sqrt{3x^2 + 2x + 16} \ .$$

6. Graph $y = \dfrac{x}{x^2-1}$ showing clearly all asymptotes and local maxima and minima.
The roots of the denominator are ± 1, so we have vertical asymptotes at $x = -1$, $x = 1$. For $x > 1$, y is positive and tends to zero as $x \to \infty$. For $x < 1, y$ is negative and tends to zero as $x \to -\infty$. For $-1 < x < 0$ we have $y > 0$ and for $0 < x < 1$, $y < 0$. This tells us about the behavior at the asymptotes. Now,

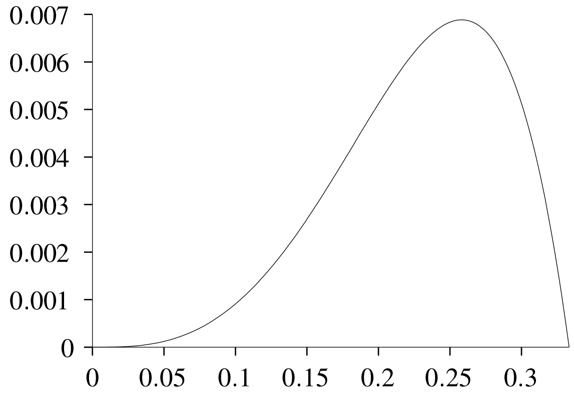
$$\frac{dy}{dx} = \frac{(x^2-1) - x(2x)}{(x^2-1)^2} = \frac{-1-x^2}{(x^2-1)^2} < 0$$

for all x, so the graph is always decreasing: there are no local maxima or minima. The graph is



7. Find the area of the region in the first quadrant bounded by the curve $y = x^3 - 9x^5$.

Answer. Draw the graph:

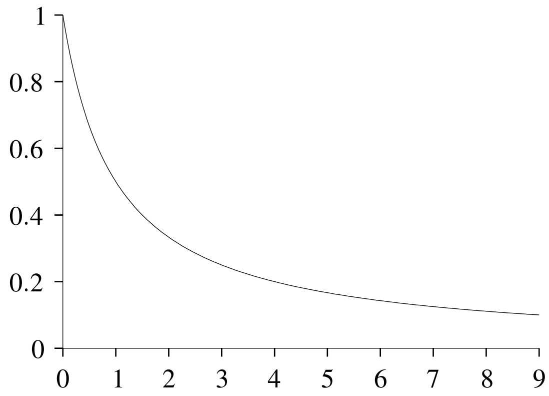


Solving the equation $x^3 - 9x^5 = 0$, or $x^3(1-9x^2) = 0$, we see that the region in the first quadrant lies between $x = 0$ and $x = 1/3$, so

$$Area = \int_0^{1/3} (x^3 - 9x^5)dx = (\frac{x^4}{4} - 9\frac{x^6}{6})\Big|_0^{1/3} = \frac{1}{3^4}(\frac{1}{4} - \frac{1}{6}) = \frac{1}{2^2 3^5} .$$

8. Find the volume of the solid obtained by rotating the curve $y = (x+1)^{-1}$, $0 \le x \le 9$ about the x-axis.

Answer. Draw the graph:



We use the method of disks: $dV = \pi y^2 dx$. This gives us

$$Volume \ = \pi \int_0^9 \frac{dx}{(x+1)^2} = \frac{-\pi}{x+1}\Big|_0^9 = -\pi(\frac{1}{10} - 1) = .9\pi \ .$$

9.The triangle with vertices at (0,0), (1,0), (0,3) is filled with an inhomogeneous muck whose density at the point (x,y) is $\delta(x,y) = 1+x$. What is the mass of the muck?

Answer. The region is that under the curve $y = 3-3x$ for x between 0 and 1. At any x we have $dMass = \delta y dx = (3-3x)(1+x)dx = 3(1-x^2)dx$. Thus

$$Mass = 3\int_0^1 (1-x^2)dx = 3(1-\frac{1}{3}) = 2 \ .$$

---

10. A cylindrical water tank, 8 ft. in diameter and 12 feet high has its base 50 feet above ground level. How much work is done in filling the tank with water? (The density of water is 62.5 lbs/ft^3).

Answer. The weight of the water of thickness dx at any level is $62.5(\pi(4^2)dx$ (since the radius of the tank is 4 ft.) The slab at height x in the tank has been raised $50+x$ feet. Thus $dW = (62.5)(16)(50+x)dx$, and the total work for the 12 ft. high tank is

$$Work = 10^3 \int_0^{12} (50+x)dx = 10^3(50x + \frac{x^2}{2})\Big|_0^{12} = 672,000 \ \ \mathrm{ft-lbs}.$$